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AN

ELEMENTARY TREATISE

ON

OPTICS,

DESIGNED FOR THE USE OF THE CADETS

OF THE

UNITED STATES' MILITARY ACADEMY.

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PREFACE.

In presenting the following pages to the public, their author advances no claims to originality. His only object has been to prepare, what appeared to him, a suitable elementary class book on the subject of which they profess to treat, for the use of the Cadets of the United States' Military Academy. In doing this, he has availed himself, principally, of the works of Mr. Coddington, Sir David Brewster, Sir John Herschell, and the Rev. Baden Powell; and to these distinguished authors, he would acknowledge his obligation for whatever of merit this little volume may be found to possess.

CONTENTS.

ELEMENTARY TREATISE

ON

OPTICS.

1. Light is that principle by whose agency we derive our sensations of external objects through the sense of sight.

2. That branch of Natural Philosophy which treats of the nature and properties of *light*, is called *Optics*.

3. All bodies are divided with respect to light, into two classes, called *Self-luminous* and *Non-luminous*.

4. Self-luminous bodies are such as possess the power of exciting light; as the sun, stars, &c. Generally, all substances become self-luminous when their temperature is sufficiently raised.

5. Non-luminous bodies are such as do not possess the power of exciting light, and are visible in consequence only of light derived from bodies of the self-luminous class.

6. Astronomical observations have shown that the communication which light produces between us and luminous objects, is not instantaneous. When the sun, for example, is in any assumed point of his orbit, the sensation of his presence there is not communicated to us till 8′ 13″ afterwards. Knowing the distance of the earth from the sun, the velocity with which light moves will easily result. It is found to be about 195,200 miles a second.

7. This extremely rapid communication is supposed to be made either by pulsations transmitted through a highly elastic fluid, as sound is transmitted through the air; or by real emanations of material particles thrown off from the surfaces of luminous bodies. Since we see objects through a certain class of bodies called *transparent*, it would follow that the pulsations or waves of the elastic fluid continue to be propagated through the interstices of these substances, or that the luminous particles, on the supposition of material emanations, pass through the same openings.

The eye admitting the free passage of light into it, the sensation of vision is supposed to arise, on the supposition of *material emanations*, from the action of the particles on certain nerves which are spread over the inner surface of the back part of that organ. In the theory of *elastic fluid*, vision is attributed to vibratory motion communicated to the same nerves by the pulsations or

waves propagated through the elastic fluid, with which they are in contact.

To give any thing like an adequate idea of the theories here referred to, would far transcend the limits of an elementary treatise; nor is it necessary to our present purpose, to know any thing with respect to the real nature of light. The investigations pursued in the following pages being founded upon data derived from actual experiment, the results will be true whatever changes may take place in the theories with respect to the *nature* of the agent or principle to which these results refer.

8. A *ray* of light, denotes any rectilinear direction in which the effect of light is conveyed. It will be convenient, hereafter, to associate with a ray, the idea of motion, and wherever this is done, nothing more is meant than a reference to the *successive* occurrences of the effect of light at the different points along the ray.

9. A collection of parallel rays, is called a *beam of light*.

10. A collection of rays diverging from, or converging to a point, is called a *pencil of light*.

11. Whatever affords a passage to light is called a *medium*, such as glass, water, air, vacuum, &c.

12. Light is transmitted from one point to another, in the same medium of homogeneous den-

sity, in right lines; for if an opaque body be interposed between the eye and an object to which it is directed, the object will be concealed from view.

Reflection and Refraction of Light.

13. When a beam or pencil of light S D, is incident upon any surface M N, (fig. 1), separating two media of different density, such as air and water, for instance, universal experience has shown that the beam or pencil will be divided into two portions, one of which will be driven back from the surface M N, in some direction as D S′, and continue in the same medium, while the other will penetrate the surface and be transmitted through the second medium in some direction as D S″. The first is said to be *reflected,* and the second *refracted.*

The circumstances attending these two portions of light being in general different, gave rise to two distinct branches of optics, *viz., Catoptrics,* and *Dioptrics;* the former treating of reflected, and the latter of refracted light. But these will be considered in connection, as by that means much time and space will be saved, and the discussion rendered general.

14. The line P D P′, (fig. 2), being supposed normal to the surface of which M N is a section by a normal plane, and the light to proceed in the direction from S to D, S D is called the *incident* ray, D S′ the *reflected ray,* and D S″ the *refrac-*

ted ray. The angle S D P is called the *angle of incidence,* P D S′ the angle of *reflection,* and P′ D S″ the *angle of refraction.*

15. All change in the direction of the incident light, either by reflection or refraction, is found to take place immediately at the surface separating the two media. If the surface of separation be curved, as M′N′ or M″N″, we may conceive a tangent plane to be drawn through the point of incidence D, when the beam of light being regarded indefinitely small, the angles of incidence, reflection and refraction will remain unchanged.

16. Experiments have shown,

First, That the incident ray, reflected ray, and refracted ray are always, except in a particular case to be noticed hereafter, *contained in the same plane normal to the surface separating the media.*

Second, That the sine of the angle of incidence is equal to that of reflection.

Third, That, for the same medium, the sine of the angle of incidence bears to the sine of the angle of refraction, a constant ratio.

If to these facts, which are the results of careful experiment, we add the rectilineal propagation of light, we shall have all the fundamental laws upon which the whole mathematical theory of Catoptrics and Dioptrics depends.

17. Denoting by φ the angle of incidence; by φ' that of refraction; and by m the constant ratio of

the sines of these angles, the third law will be expressed by the equation,

$$\sin \varphi = m \sin \varphi' \quad . \quad . \quad . \quad (1)$$

The angle which any ray makes with the normal, will be estimated from that part of the normal lying in the medium with the ray, and in a direction towards the incident ray, from the part of the normal nearest to it. Thus, (fig. 3), the angle which the reflected ray D S′ makes with the normal, is equal to 360°—P D S′ ; while the angle made by the refracted ray D S″, is P′D S″, being estimated from D P′, in the same direction around D. By this convention, we shall be able to convert all expressions relating to refraction into others appertaining to reflection, by simply changing m into -1. This in equation (1), gives

$$\sin \varphi = - \sin \varphi' \quad . \quad . \quad . \quad (2)$$

which expresses the second law; or, to include both reflection and refraction under the same formula,

$$\sin \varphi = \pm\, m \sin \varphi'.$$

18. M′N′ (fig. 4), being a section of the separating surface, M N that of a tangent plane at the point of incidence D, P D P′ the normal, S D the incident, D S′ the reflected, and S D″ the refracted rays ; the angle S′ D Q, made by the reflected ray with its direction before incidence, is called the *deviation by reflection,* and S″DQ, the *deviation by refraction;* the rays are said to be

deviated at the point D; and the surface of which M′ N′ is a section, is called the *deviating surface.*

19. The numerical value of m which expresses the quotient arising from dividing the sine of incidence by the sine of refraction, although constant for the same medium, varies from one medium to another. As a general rule, it is *greater* than unity when light passes from any medium to another of greater density, such as from air to water, from water to glass, &c.; and *less* than unity when light passes from any medium to one less dense, as from water into air.

There is a remarkable exception to this rule in the case of combustible substances, these always refracting more than other substances of the same density.

From what has been said, it is obvious that a ray of light on leaving any medium and entering one more dense, will be bent towards the normal to the deviating surface, while the reverse will be the case when the medium into which the ray passes is less dense than the other.

The numerical value of m, has been determined for a great variety of substances, solids liquids and gases, on the supposition of the deviating surface being that which separates the various substances considered from a vacuum. If all bodies possessed equal density, the value of m, or the index of refraction, might be taken as the measure of the refractive power of the substance to which it belongs,

but this not being the case, Sir Isaac Newton has shown, that on the supposition of the *law according to which all substances act upon light being of the same form,* the refractive power will be proportional to the excess of the square of the index above unity, divided by the specific gravity. Calling n the absolute refractive power, m, the index of refraction, S, the specific gravity, and A, a constant coefficient, we shall have according to this rule,

$$n = A.\frac{m^2-1}{S} \quad . \quad . \quad . \quad (3)$$

The following table shows the value of m, and n, for the different substances named, the value of m being taken on the passage of light from a vacuum.

Table of Refractive indices and Refractive Powers.

Substances.	m	$n=\frac{m^2-1}{S}$
Chromate of Lead,	2.97 2.50	1.0436
Realgar,	2.55	1.666
Diamond,	2.45	1.4566
Glass-flint,	1.57	0.7986
Glass Crown,	1.52	
Oil of Cassia,	1.63	1.3308
Oil of Olives,	1.47	1.2607
Quartz,	1.54	0.5415
Muriatic Acid,	1.40	
Water,	1.33	0.7845
Ice,	1.30	
Hydrogen,	1.000138	3.0953
Oxygen,	1.000272	0.3799
Atmospheric Air,	1.000294	0.4528

On the Deviation of Light at Plane Surfaces.

20. Let M N, (fig. 5), be a deviating surface, separating any medium *B* from a vacuum *A*. A ray of light S D, being incident at D, will be deviated according to the law expressed by equation (1).

$$\sin\varphi = m \sin\varphi'$$

m being the index of refraction of the medium *B*. The refracted ray D D′, meeting a second surface M′ N′, parallel to the first, and passing again into a vacuum, will be refracted so as to satisfy the equation,

$$\sin\varphi' = m' \sin\varphi''$$

the angle of incidence on the second surface being the same as that of refraction at the first, and denoting by m' the index of refraction from the medium *B* to the vacuum. But, in this case, experiment gives,

$$m' = \frac{1}{m}$$

whence we obtain, by means of the foregoing equations,

$$\sin\varphi = \sin\varphi''$$

that is, the ray after passing a medium bounded by parallel plane faces, is not deviated, but remains parallel to its first direction.

The ray $D'' D'''$, being supposed to traverse a second medium bounded by plane parallel faces, and of which the refractive index is m'', will undergo no deviation; and the same may be said of any number of media bounded by similar faces. If, now, the spaces between the media be diminished indefinitely so as to bring them into actual contact, experiment shows there will still be no deviation, which might have been inferred.

21. Let us next suppose, (fig. 6), a ray to traverse two media, bounded by plane parallel faces, the media being in contact, and having their refractive indices denoted by m and m'; we shall have by calling m'', the index of refraction of the second, or denser medium in reference to the first,

$$\begin{aligned} \sin\varphi &= m \sin\varphi' \\ \sin\varphi' &= m'' \sin\varphi'' \quad . \quad . \quad . \quad (4) \\ \sin\varphi'' &= \frac{1}{m'} \sin\varphi. \end{aligned}$$

Multiplying these equations together, there will result

$$m'' = \frac{m'}{m},$$

That is, *to find the index of refraction where a ray*

passes from any one medium to another, divide the index of the second by that of the first, referred to a vacuum.

22. If a ray pass from a medium to another more dense (fig. 7), the index m'' will be greater than unity, and from equation (4), we shall have

$$\sin \varphi' > \sin \varphi'',$$

and if $\sin \varphi'$ be taken a maximum, or the angle of incidence be 90°, equation (4) will give,

$$\frac{1}{m''} = \sin \varphi'' \quad . \quad . \quad . \quad (5)$$

from which results a maximum limit of the angle of refraction. If m'' be taken equal to $1{\cdot}52$ for the atmosphere and crown glass,

$$\sin \varphi'' = {\cdot}657,$$

or

$$\varphi'' = 41^\circ\ 5'\ 30'' \text{ nearly};$$

for air and water, $m'' = 1{\cdot}33$, and

$$\varphi'' = 48^\circ.\ 15';$$

that is to say, the greatest angle of refraction that can exist when light passes from air into crown glass, is $41^\circ.\ 5'.\ 30''$; and from air into water, $48^\circ\ 15'$.

If the ray pass from a medium to another less dense, (fig. 8), m'' will be less than unity, and equal to the reciprocal of its former value; equation (4) will then give,

$$\sin \varphi'' > \sin \varphi'$$

taking the maximum value for $\sin \varphi'' = 1$, we shall obtain from same equation,

$$\sin \varphi' = \frac{1}{m''}$$

this value of the sine of the angle of incidence, which corresponds to the greatest angle of refraction, when light passes from any medium to one less dense, is the same as that found before for the greatest angle of refraction when the incidence was taken a maximum in the passage of light from one medium to another of greater density.

This value in the case of air and glass, is ·657; corresponding to an angle of 41°. 5′. 30″; and for air and water, the angle is 48°. 15′.

If the angle φ' be taken greater than that whose sine is $\frac{1}{m''}$, the angle of refraction, or emergence from the denser medium, will be imaginary, and the light will be *wholly reflected* at the deviating surface. This maximum value for φ', is called the *angle of total reflection*. Light cannot, therefore, pass out of crown glass into air under a greater angle of incidence than 41°. 5′. 30″; nor

out of water into air under a greater angle than 48°. 15′.

The *maximum limit of refraction*, and the case of *total reflection*, are attended with many interesting results. If an eye be placed in a more refracting medium than the atmosphere, as that of a fish under water, it will perceive, by the limit of refraction, all objects in the horizon elevated in the air, and brought within 48°. 15′ of the zenith, while objects in the water would appear to occupy the belt included between this limit and the horizon by total reflection.

Those remarkable cases of *mirage*, where objects are seen suspended in the air, and oftentimes inverted, are explained by ordinary refraction and total reflection. It is well known that in the ordinary state of the atmosphere, its density decreases as we ascend; a ray of light, therefore, entering near the horizon, would undergo a series of refractions, and reach the eye with an increased inclination to the surface of the earth; or would appear to come from a point in the heavens above that occupied by a body from which it proceeded. Hence, the effect of the atmosphere is to increase the apparent altitude of all the heavenly bodies.

Dr. Wollaston suggested a method, founded on the limit of *total reflection*, to determine the refractive powers of different substances. If the angle of incidence φ', be measured by any device, equation (1) will give,

$$m = \frac{1}{\sin \varphi'}.$$

from which the absolute refractive power may be deduced by equation (3).

23. The deviating surfaces have, thus far, been supposed parallel. If they be inclined to each other, as M N, M N′, (fig. 9), we shall have what is called an *optical prism*, which consists of any refracting substance bounded by plane surfaces intersecting each other.

M N and M N′ are called the deviating planes, and the angle under which they are inclined, is called the *refracting angle* of the prism.

24. *Required the deviation of a ray of light on passing through a prism.*

Let S D (fig. 10), be the incident, D D′ the first, and D′ S′ the second refracted rays. The total deviation will be S E S″, which will be called δ; calling the refracting angle of the prism α, and adopting the notation of the figure, we shall have

$$\delta = \text{E D D}' + \text{E D}' \text{D} = \varphi - \varphi' + \psi - \psi' = \varphi + \psi - (\varphi' + \psi')$$

$$180^\circ \text{ or } \pi = \alpha + \text{M D D}' + \text{M D}' \text{D} = \alpha + \frac{\pi}{2} - \varphi' + \frac{\pi}{2} - \psi'$$

or

$$\alpha = \psi' + \varphi' \quad . \quad . \quad . \quad . \quad (6)$$

hence

$$\delta = \varphi + \psi - \alpha \quad . \quad . \quad . \quad (7)$$

The deviation of a ray of light on passing through a prism is, therefore, equal to the *sum of*

the angles of incidence and emergence, diminished by the refracting angle of the prism.

The refracting angle for the same prism being constant, the deviation will depend upon the angles of incidence and emergence.

25. *Required the relation between the angles of incidence and emergence in order that the deviation shall be a minimum.*

For this purpose we have equations

$$\delta = \varphi + \psi - \alpha \quad . \quad . \quad . \quad . \quad (7)$$
$$\alpha = \psi' + \varphi' \quad . \quad . \quad . \quad . \quad . \quad (6)$$
$$\sin\varphi = m \sin\varphi' \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$\sin\psi = m \sin\psi' \quad . \quad . \quad . \quad . \quad . \quad (1)'$$

From equation (7), we have

$$d\,.\,\delta = d\,\varphi + d\,\psi = \left(\frac{d\,\psi}{d\,\varphi} + 1\right) d\,\varphi = 0$$

or

$$\frac{d\,\psi}{d\,\varphi} + 1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

from equations (1) and (1)′,

$$\frac{d\,\psi}{d\,\varphi} = \frac{\cos\varphi,\, \cos\psi'}{\cos\varphi' \cos\psi} \cdot \frac{d\,\psi'}{d\,\varphi'} \quad . \quad . \quad . \quad . \quad (b)$$

From equation (6)

$$\frac{d\,\psi'}{d\,\varphi'} = -1$$

this last combined with (b), and (a), we get

$$\frac{\cos\varphi \,.\, \cos\psi'}{\cos\varphi' \,.\, \cos\psi} = 1$$

which will be satisfied on making $\varphi = \psi$, and $\varphi' = \psi'$.

Hence, the *deviation is a minimum, when the angle of incidence is equal to that of emergence.* This supposition being made in equations (7) and (6), they give

$$\varphi = \tfrac{1}{2}(\alpha + \delta)$$
$$\varphi' = \tfrac{1}{2}\alpha$$

and these values in equation (1), give,

$$m = \frac{\sin\frac{1}{2}(\alpha + \delta)}{\sin\frac{1}{2}\alpha} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

we have, therefore, only to measure the deviation when a minimum, to find the index of refraction of the medium of which the prism is made, supposing its refracting angle known.

This furnishes one of the best methods by which the refractive power of different substances may be found. If the substance be a liquid, we may unite two plane glasses, making any angle with each other, by means of a little cement placed along their edges, and place the liquid between them where it will be held in sufficient quantity by capillary attraction.

When the ray is incident at right angles upon the first surface, we have,

$$\varphi = 0$$
$$\varphi' = 0$$

and from equations (7) and (6), there result,

$$\delta = \psi - \alpha,$$
$$\alpha = \psi'$$

whence

$$\sin(a + \delta) = m \sin a \quad . \quad . \quad . \quad (8)'.$$

Deviation at plane surfaces by refraction, will be again referred to in a subsequent part of the text.

26. Let M N, M N′ (fig. 11), be two plane reflectors, meeting in a line projected in M ; S D, a ray incident at the point D, and contained in a plane perpendicular to the intersection of the reflectors; it will be deviated at the point D of the first reflector, again at the point D′ of the second, and so on.

Required the circumstances attending these deviations.

Call the first angle of incidence,	φ_1
second, . . .	φ_2
third, . . .	φ_3
&c.,	
n^{th}, . . .	φ_n

In the triangle P D D′, the angle at P is equal

to the inclination of the reflectors, which we will call i, and we shall have

$$\left.\begin{aligned}\varphi_1 - \varphi_2 &= i\\ \varphi_2 - \varphi_3 &= i\\ \varphi_3 - \varphi_4 &= i\\ &\cdots\\ \varphi_{\overline{n-2}} - \varphi_{\overline{n-1}} &= i\\ \varphi_{\overline{n-1}} - \varphi_n &= i\end{aligned}\right\} \quad \ldots \quad (a)$$

$$\varphi_1 - \varphi_n = \overline{n-1}\,.\,i$$

$$\varphi_n = \varphi_1 - \overline{n-1}\,.\,i \quad . \quad (9)$$

If φ_1 be any multiple of i, as $\overline{n-1}\,.\,i$,

$$\varphi_1 - \overline{n-1}\,.\,i = 0, \quad \ldots \quad (10)$$

or the n^{th} incidence will be perpendicular to the reflector, and the ray will consequently return upon itself.

Example 1*st*. Suppose the angle made by the reflectors to be 6°, and the first angle of incidence, or $\varphi_1 = 60°$, required the number of reflections before the ray retraces its course.

These values in equation (10), give,

$$60° - \overline{n-1}\,.\,6° = 0$$

or

$$\text{ans:} \quad n = 11$$

Example 2*d*. The angle of the reflectors being

15°, and the first angle of incidence 80°, required the *fourth* angle of incidence;

These values in equation (9), give

$$\varphi_4 = 80^\circ. - \overline{4-1}.15^\circ.$$
$$\text{ans:} \quad \varphi_4 = 35^\circ$$

If φ_1 be not a multiple of i, there will be some value for n that will make $\overline{n-1}.i$, greater than φ_1, in which case, $\varphi_1 - \overline{n-1}.i$ will be negative; that is, at the n^{th} incidence, the ray will be on the opposite side of the perpendicular. It will therefore return, but not, as before, by the same path.

Example 3d. The angle of the reflectors being 7°, the first angle of incidence 69°, required the number of reflections before the ray returns, and the first angle of incidence of the returning ray. These values in equation (9) reduce it to

$$\varphi_n = 69^\circ - \overline{n-1}.7^\circ = 76^\circ - 7n$$

If $n = 10$,

$$\varphi_n = 76^\circ - 70^\circ = 6^\circ$$

If $n = 11$,

$$\varphi_n = 76^\circ - 77^\circ = -1^\circ$$

or the ray begins to return at the eleventh incidence and the value of the angle of incidence is 1°.

It is obvious that the angle of incidence of the returning ray will increase at every deviation; there will, therefore, be some value of the increased angle which will either be equal to 90°; in which case, the ray will finally be reflected by

one of the reflectors into a direction parallel to the other; or greater than 90° when the ray would meet the other reflector on being produced back.

Adding the first two equations in group (*a*), we have

$$\varphi_1 - \varphi_3 = 2i$$

or

$$SS'D = 2i$$

That is, the angle made by the incident ray and the same ray after two reflections, is equal to double the angle of the reflectors. It follows, therefore, that if the angle of the reflectors be increased or diminished by giving motion to one of the reflectors, the angular velocity of the reflected ray will be double that of the reflector. This is the principle upon which reflecting instruments for the measurement of angles are constructed.

Of the Deviation of Rays at Sperical Surfaces.

27. Let M D O N (fig. 12), be a section of a spherical surface separating two media of different densities, as air and glass, having its centre at C, on the line O C, which will be called the axis of the deviating surface; f D a ray of light, incident at D, and D S, the direction of this ray after deviation, which being produced back will intersect the axis at f'. The point O, where the axis meets the surface, is called the *vertex*. Call f D,

u; f' D, u'; C D, r; O f', f'; O f, f; and the angle O C D, θ.

In the triangle C D f, we have the relation,

$$\frac{\sin \varphi}{\sin \theta} = \frac{f - r}{u}.$$

and in the triangle C D f',

$$\frac{\sin \theta}{\sin \varphi'} = \frac{u'}{f' - r}.$$

these combined with,

$$\sin \varphi = m \sin \varphi',$$

we have,

$$m u . (f' - r) = u' (f - r) \quad . \quad . \quad (11)$$

The first of these triangles will also give,

$$u^2 = (f - r)^2 + r^2 + 2 (f - r) . r . \cos \theta$$

and the second,

$$u'^2 = (f' - r)^2 + r^2 + 2 (f' - r) . r . \cos \theta.$$

these latter equations by reduction become,

$$u^2 = f^2 - 2 r (f - r). \text{versin.}\ \theta$$
$$u'^2 = f'^2 - 2 r (f' - r). \text{versin.}\ \theta.$$

Denoting the versin. θ, by z, and eliminating

u and u', between these equations and equation (11), there will result,

$$(f-r).\sqrt{f'^2-2r(f'-r).z}=m(f'-r).\sqrt{f^2-2r(f-r).z} \quad (12)$$

This is a general equation for finding the intersection of deviated rays with the axis. The relation between f and f', is somewhat complicated, and it is obvious that if f, be taken constant, the value of f' will vary for different values of θ; that is to say, *if a pencil of rays proceed from a point on the axis, they will, after deviation, meet the axis in different points, depending upon the distance of the point of incidence from the vertex.*

Small Direct Pencil.

28. A pencil of light having its central ray coincident with the axis of the deviating surface, is called *a direct pencil;* and if such a pencil be taken very small, the quantity z, in equation (12), will be so small that the products of which it is a factor, may, without material error, be omitted. This will reduce equation (12) to

$$(f-r).f'=m.(f'-r).f$$

or

$$f'=\frac{mrf}{(m-1).f+r} \quad . \quad . \quad . \quad (13)$$

and taking the reciprocal,

$$\frac{1}{f'} = \frac{m-1}{m r} + \frac{1}{m . f} \quad . \quad . \quad . \quad . \qquad (14)$$

If f be constant, or the rays all proceed from the same point on the axis, f' will also be constant for the same medium, and curvature; hence, *all the rays after deviation will meet in the same point on the axis.* This point is called *the focus*, and the point from which the rays first proceeded, *the radiant;* and because of the mutual dependence of these points upon each other with respect to their position, they are often called *conjugate foci.* If the rays after deviation only meet the axis by being produced back, the focus becomes imaginary, and is said to be *virtual;* the radiant is also said to be virtual when the rays converge to a point before deviation.

If we suppose the ray deviated at the first, to be incident on a second surface M′ N′, (fig. 13), having a radius r', and at a distance t, from the first, measured on the axis, we may regard this ray as proceeding originally from f'; and estimating the distance from the vertex O′ to the point at which the ray, after deviation at the second surface, meets the axis, we shall have, from equation (14), by calling this distance f'', and the index at the second surface m',

$$\frac{1}{f''} = \frac{m'-1}{m' r'} + \frac{1}{m'(f'+t)} \quad . \quad . \quad . \qquad (15)$$

The value of $(f'+t)$, deduced from (14), and substituted in equation (15), gives a direct relation between f and f'', in functions of r, r', m, m' and t.

For a *third* surface, as M'' N'' (fig. 14), we shall, in like manner, have

$$\frac{1}{f'''}=\frac{m''-1}{m''r''}+\frac{1}{m''(f''+t+t')} \quad . \quad . \ (16)$$

If in this equation, we substitute the value of $(f''+t+t')$, deduced from the equation derived from (14) and (15), by eliminating f', we shall have the relation between f and f''', expressed in functions of $r, r', r'', m, m', m'', t, t'$; and so on, of any number of surfaces.

In the same way (fig. 15),

$$\frac{1}{f''''}=\frac{m'''-1}{m'''r'''}+\frac{1}{m'''(f'''+t+t'+t'')} \quad . \ (17)$$

28. From equation (13), we get,

$$f'+t=\frac{mrf+(\overline{m-1}.f+r)\,t}{\overline{m-1}.f+r}$$

this substituted in equation (15), at same time making $m'=\dfrac{1}{m}$, which is supposing the ray to

pass into the first medium after having traversed the medium bounded by the two deviating surfaces, that equation reduces to,

$$\frac{1}{f''} = \frac{1-m}{r'} + \frac{m\,(\overline{m-1}\,.f+r)}{m\,rf+(f.\overline{m-1}+r)\,t} \quad . \quad (18)$$

The same process being continued with respect to rays deviated at three, four, &c., surfaces, we would obviously obtain a series of equations expressive of relations between f''' and f, f^{iv} and f, f^{v} and f, &c.; the equations becoming more and more complex in proportion as we consider a great number of deviating surfaces. They will be greatly simplified, however, by neglecting the consideration of the thickness, which may be done in almost all cases of practice without incurring much error. Dropping the thickness, equation (18) will become, by retaining m',

$$\frac{1}{f''} = \frac{m'-1}{m'r'} + \frac{1}{m'}\left\{\frac{m-1}{m\,r} + \frac{1}{m\,f}\right\};$$

the value of $\frac{1}{f''}$, substituted in equation (16), gives,

$$\frac{1}{f'''} = \frac{m''-1}{m''\,r''} + \frac{1}{m''}\left\{\frac{m'-1}{m'\,r'} + \frac{1}{m'}\left(\frac{m-1}{m\,r} + \frac{1}{m\,f}\right)\right\}. \ (16)'$$

and the value of $\frac{1}{f'''}$, in equation (17), gives,

$$\frac{1}{f''''} = \frac{m'''-1}{m''' r'''} + \frac{1}{m'''}\left[\frac{m''-1}{m'' r''} + \frac{1}{m''}\left\{\frac{m'-1}{m' r'} + \frac{1}{m'}\left(\frac{m-1}{m r} + \frac{1}{mf}\right)\right\}\right] \quad (17)'$$

and so for additional surfaces.

If we now suppose the medium between the *second* and *third*, *fourth* and *fifth*, *sixth* and *seventh*, *&c.*, deviating surfaces, the same as that in which the light moved before the first deviation, we shall have the case of a number of refracting media bounded by spherical surfaces, situated in a homogeneous medium, such as the atmosphere, for example, and nearly in contact. Hence,

$$m = \frac{1}{m'}; \quad m'' = \frac{1}{m'''}; \quad m'''' = \frac{1}{m'''''}, \text{ \&c.}$$

and the foregoing equations reduce to,

$$\frac{1}{f''} = (m-1) \cdot \left\{\frac{1}{r} - \frac{1}{r'}\right\} + \frac{1}{f} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$\frac{1}{f'''} = \frac{m''-1}{m'' r''} + \frac{1}{m''}\left\{\overline{m-1} \cdot \left(\frac{1}{r} - \frac{1}{r'}\right) + \frac{1}{f}\right\} \quad . \quad . \quad (20)$$

$$\frac{1}{f''''} = \overline{m''-1} \cdot \left(\frac{1}{r''} - \frac{1}{r'''}\right) + \overline{m-1} \cdot \left(\frac{1}{r} - \frac{1}{r'}\right) + \frac{1}{f} \quad (21)$$

&c., &c.

29. Any medium bounded by curved surfaces,

used for the purpose of deviating light by refraction, is called a *lens*. Equation (19) relates, therefore, to the deviation of a small pencil of light by a single spherical lens; f, denoting the distance of the radiant, and f'', that of the focus from the lens. Equation (20), relates to the refraction or deviation by a single lens and a second medium of indefinite extent bounded on one side by a spherical surface nearly in contact with the lens. Equation (21), relates to deviation by two spherical lenses close together, f and f'''' denoting, as before, the radiant and focal distances.

30. If the rays be parallel before the first deviation, f will be infinite, or $\frac{1}{f} = 0$, and equations (19), (20), and (21), will reduce to

$$\frac{1}{f''} = \overline{m-1} \cdot \left(\frac{1}{r} - \frac{1}{r'}\right)$$

$$\frac{1}{f'''} = \frac{m''-1}{m''r''} + \frac{1}{m''}\left[\overline{m-1} \cdot \left(\frac{1}{r} - \frac{1}{r'}\right)\right]$$

$$\frac{1}{f''''} = \overline{m''-1} \cdot \left(\frac{1}{r''} - \frac{1}{r'''}\right) + \overline{m-1} \cdot \left(\frac{1}{r} - \frac{1}{r'}\right)$$

&c., &c.

The values of f'', f''', f'''', &c., deduced from these equations are called the *principal focal distances*, being the *focal* distances for parallel rays. Denoting these distances by $F_{,,}$, $F_{,,,}$, $F_{,,,,}$, &c., and $\left(\frac{1}{r} - \frac{1}{r'}\right)$, $\left(\frac{1}{r''} - \frac{1}{r'''}\right)$ &c., by $\frac{1}{\varrho}$, $\frac{1}{\varrho''}$, $\frac{1}{\varrho''''}$ &c., we shall

have by including equation (14), the following table:

$$\left.\begin{aligned}
\frac{1}{F_{\prime}} &= \frac{m-1}{m\,r} \\
\frac{1}{F_{\prime\prime}} &= \frac{m-1}{\varrho} \\
\frac{1}{F_{\prime\prime\prime}} &= \frac{m''-1}{m''\,r''} + \frac{1}{m''}\left(\frac{m-1}{\varrho}\right) \\
\frac{1}{F_{\prime\prime\prime\prime}} &= \frac{m''-1}{\varrho''} + \frac{m-1}{\varrho} \\
\frac{1}{F_{\prime\prime\prime\prime\prime}} &= \frac{m''''-1}{m''''\,r''''} + \frac{1}{m''''}\left(\frac{m''-1}{\varrho''} + \frac{m-1}{\varrho}\right) \\
\frac{1}{F_{\prime\prime\prime\prime\prime\prime}} &= \frac{m''''-1}{\varrho''''} + \frac{m''-1}{\varrho''} + \frac{m-1}{\varrho} \\
&\text{\&c., \&c., \&c.}
\end{aligned}\right\} \quad \ldots \quad \text{(A)}$$

An examination of the alternate formulas of the above table leads to this result, viz. *that the reciprocal of the principal focal distance of any combination of lenses, is equal to the sum of the reciprocals of the principal focal distances of the lenses taken separately;* which may be expressed in a general way by the equation,

$$\frac{1}{F} = \Sigma\left(\frac{1}{F}\right) \quad \ldots\ldots \quad (22)$$

wherein $\left(\frac{1}{F}\right)$, denotes the reciprocal of the

principal focal distance of any one lens in the combination Σ, that the algebraic sum of these is to be taken, and $\frac{1}{F}$, the reciprocal for the combination.

Substituting the first member of the first equation, in group (A), and the first members of the alternate equations beginning with the second, for their corresponding values in equations (14), (19), (21), &c., we finally obtain,

$$\frac{1}{f'} = \frac{1}{F_{,}} + \frac{1}{mf} \quad \ldots\ldots \quad (23)'$$

$$\frac{1}{f''} = \frac{1}{F_{,,}} + \frac{1}{f} \quad \ldots\ldots \quad (23)$$

$$\frac{1}{f''''} = \frac{1}{F_{,,,,}} + \frac{1}{f} \quad \ldots\ldots \quad (24)$$

$$\frac{1}{f''''''} = \frac{1}{F_{,,,,,,}} + \frac{1}{f} \quad \ldots\ldots \quad (25)$$

Equations (23), (24), and (25), are of a convenient form for discussing the circumstances attending the deviation of light by refraction through a single lens, or a combination of lenses placed close together ; and equation (23)′, the deviation by reflection at a single surface.

31. The several terms of these equations are the reciprocals of elements involved in the discussions which are to follow. But, the pencil of light being small, the versed sine of the arc O D, (fig. 16), may be disregarded, and this arc may be con-

sidered as coinciding with the tangent at the vertex O, and as having been described about either of the points C, V, or U, as a centre, indifferently, hence

$$\text{D V O} : \text{D U O} :: \frac{1}{f_{\prime}} : \frac{1}{f},$$

that is, the relative *divergence* or *convergence* of the incident and deviated pencils will be expressed by the reciprocals of the conjugate focal distances f and f'.

32. *The power of a lens is its greater or less capacity to deviate the rays that pass through it.*

In equations (23), (24), (25), &c., $\frac{1}{F_{\prime\prime}}$, $\frac{1}{F_{\prime\prime\prime}}$, $\frac{1}{F_{\prime\prime\prime\prime\prime\prime}}$ &c., will measure the divergency or convergency of parallel rays after deviation; and as these measures are expressed in functions of the indices of refraction and $\frac{1}{\varrho}$, or $\left(\frac{1}{r} - \frac{1}{r'}\right)$, &c., they will be constant for the same media and curvature; and hence become terms of comparison for the other two terms which enter into the equations to which they respectively belong.

From what has been said, it is apparent that $\frac{1}{F}$, in equation (22), will measure the degree of convergence or divergence of parallel rays after deviation by any combination of spherical lenses what-

ever, and will consequently be the measure of the *power of the combination;* and as $\left(\frac{1}{F}\right)$, is the measure of the power of any one lens of the combination, we have this rule for finding the power of any system of lenses, viz: *Find the power of each lens separately, and take the Algebraic sum of the whole.*

33. It may be convenient to express the relation in equations (23)′, (23), (24), &c., by referring to the centre of curvature of the deviating surfaces as an origin. For this purpose, let O D, (fig. 17), be a section of the deviating surface; denoting the distances of the radiant and focal points from the centre C, by c and c', respectively, we have by inspection,

$$f = r + c,$$
$$f' = r + c',$$

which in equation (13), give, after reduction,

$$\frac{1}{c'} = \frac{m-1}{r} + \frac{m}{c} \quad . \quad . \quad . \quad . \quad (26)$$

and for a second deviating surface whose centre of curvature is at a distance t, from that of the first, we obtain from equation (26),

$$\frac{1}{c''} = \frac{m'-1}{r'} + \frac{m'}{c'-t} \quad . \quad . \quad . \quad (27)$$

and for a third, whose centre is at a distance $t,'$ from that of the second,

$$\frac{1}{c'''} = \frac{m''-1}{r''} + \frac{m''}{c''-t-t'} \quad . \quad . \ (28)$$

&c., &c.,

c' being eliminated between (26) and (27), a relation between c and c'' will result ; in like manner, c'' being made to disappear by means of this derived equation and equation (28), there will result an equation in terms of c''' and c, and so for any others.

Application of the preceding theory to the deviation of light by refraction through the various kinds of spherical lenses.

34. A lens has been defined to be, any medium bounded by curved surfaces, used for the purpose of deviating light by refraction ; the surfaces are generally spherical.

A, (fig. 18), called a *double convex* lens, is formed by two spherical surfaces, having their centres on opposite sides of the lens. When the curvature of the two surfaces is the same, the lens is said to be equally convex.

B, is a lens with one of its faces plane, the other spherical, and is called a *plano-convex lens*.

C, is a *double concave lens*, having the centres of its curved surfaces on opposite sides.

D, is a *plano-concave lens*, having one face plane and the other concave.

E, has one face concave, and the other convex the convex face having the greatest curvature ; this lens is called a *meniscus*.

F, like the meniscus, has one face concave and the other convex, but the concave face has the greatest curvature ; this is called a *concavo-convex lens*.

The line *U V*, containing the centres of the spherical surfaces, is called the axis.

35. A moment's consideration will show that all the circumstances of convergence or divergence, attending the deviation of light by any one of these lenses, will be made known by equation(23), it being only necessary to note the different cases arising out of the various combinations of surfaces by which the lenses are formed ; these cases depend on the *signs* of the *radii*.

Equations (23), (24), and (25), &c., were deduced on the supposition that r is positive when the concave side of the surface is turned towards incident light ; it will, of course, be negative when the convex side is turned in the same direction. Besides, f was taken positive for a *real radiant*, or when the rays are supposed to diverge from any point upon the axis of the lens, before deviation ; on the contrary, it will become negative, when the rays are received by the deviating surface, in a state of convergence to a point behind the lens. The signs of f', f'', &c., have been taken positive when the deviated rays meet the axis on being produced back. The foci are then virtual. When

the rays meet the axis on the opposite side of the lens or lenses, f', f'', &c., become negative, and will correspond to *real* foci.

The several lenses may be described as follows:

1	*Double Convex*, . . .		$-r$ and $+r'$
2	*Plano-Convex*, convex side to incident light, . .		$-r$ and $+r'=\infty$
	Do. plane side to incident light, . .		$+r=\infty$ and $+r'$
3	*Meniscus*, convex side turned to incident light, .	$r<r'$,	$-r, -r'$
	Same, concave side do. do.	$r>r'$	$+r, +r'$
4	*Double Concave*, . . .		$+r, -r'$
5	*Plano-Concave*, concave side to incident light, . .		$+r, +r'=\infty$
	Same, plane side to do do		$+r, =\infty$, and $-r'$
6	*Concavo-Convex*, concave side to incident light,	$r<r'$	$+r, +r'$
	Same, reversed,	$r>r'$	$-r, -r'$

36. For a double convex lens, (fig. 19),

$$\frac{1}{f''}=\frac{1}{F_{\prime\prime}}+\frac{1}{f} \quad . \quad . \quad . \quad . \quad . \quad (23)''$$

$$\frac{1}{F_{\prime\prime}}=\frac{m-1}{\rho}=-\overline{m-1}\left(\frac{1}{r}+\frac{1}{r'}\right),$$

and as long as $m>1$ we shall have,

$$\frac{1}{f''}=-\frac{1}{F_{\prime\prime}}+\frac{1}{f} \quad . \quad . \quad . \quad . \quad . \quad (23)''$$

For $\frac{1}{F_{,,}} > \frac{1}{f}$, or $f > F_{,,}$, f'' will be negative; or the focus will be real, and the rays will converge after deviation.

For $\frac{1}{F_{,,}} < \frac{1}{f}$, or $f < F_{,,}$, f'' will be positive, the focus virtual, and the rays will diverge after deviation.

If $\frac{1}{F_{,,}} = \frac{1}{f}$, or $F_{,,} = f$; $\frac{1}{f''} = 0$, or $f'' =$ infinity, and the rays, after deviation, will be parallel.

If the rays be received by the lens in a state of convergence, $\frac{1}{f}$ or f, will be negative, and,

$$\frac{1}{f''} = -\left(\frac{1}{F_{,,}} + \frac{1}{f}\right)$$

or the focus will always be real.

If the rays diverge from a point at a distance from the lens equal to double the principal focal distance,

$$\frac{1}{f''} = -\frac{1}{F_{,,}} + \frac{1}{2F_{,,}}$$

or

$$f'' = -2F_{,,}.$$

For all cases of diverging light we have,

$$\frac{1}{f''} < \frac{1}{f}$$

or, (31), the rays will diverge less after deviation; and for all cases of converging light we shall have,

$$\frac{1}{f''} > \frac{1}{f'};$$

or the rays will converge more after deviation. Hence, the effect of a double convex lens is to *collect* the rays.

The focal distance for this lens in the case of diverging rays in functions of m, r, r' and f, is

$$f'' = \frac{f \,.\, r \,.\, r'}{r r' - f(m-1)(r+r')}.$$

If the lens be supposed of glass, $m = \frac{3}{2}$, nearly, and

$$f'' = -\frac{2 f r r'}{f.(r+r') - 2 r r'}.$$

If the lens be equally convex, $r = r'$, and

$$f'' = -\frac{f.r}{f-r};$$

and if the rays be supposed parallel before deviation, $f =$ infinity, and

$$f'' = -r.$$

37. A like discussion may be gone through with each of the other lenses described. This being done, the results will conform to those exhibited in the following *table.*

Lens.	Incident pencil.	$\frac{1}{f''}$	Sign of f''.		Refrac. pencil.
Convex $-F_{\prime\prime}$	Diverging $+f$	$-\frac{1}{F_{\prime\prime}}+\frac{1}{f}$	$f > F_{\prime\prime}$	$-$	Converges
			$f < F_{\prime\prime}$	$+$ $f'' > f$	Diverges less.
	Converging $-f$	$-\frac{1}{F_{\prime\prime}}-\frac{1}{f}$		$-$ $f'' < f$	Converges more.
Concave $+F_{\prime\prime}$	Diverging $+f$	$\frac{1}{F_{\prime\prime}}+\frac{1}{f}$		$+$ $f'' < f$	Diverges more.
	Converging $-f$	$\frac{1}{F_{\prime\prime}}-\frac{1}{f}$	$f > F_{\prime\prime}$	$+$	Diverges.
			$f < F_{\prime\prime}$	$-$ $f'' > f$	Converges less.

A similar table may also be constructed by formula (24), for a combination of any of the spherical lenses taken two and two, and by formula (25), for any combination taken three and three, and so on.

In general, it may be inferred from the preceding table, that convex lenses tend to collect the incident rays, while concave lenses, on the contrary, tend to disperse them.

38. Differentiating equation (23), we have

$$-\frac{df''}{f''^2} = \pm\frac{df}{f^2};$$

the upper sign answers to the case where f'' and f, have different signs, and the lower to that in which the signs are the same; which shows that when the conjugate foci are in motion, they move in the same direction.

39. If the lens be a sphere, $m' = \frac{1}{m}$, in equation (27), and $t = 0$; and eliminating c', by means of equation (26), we obtain

$$\frac{1}{c''} = \frac{2(m-1)}{mr} + \frac{1}{c} \quad . \quad . \quad . \quad (29).$$

40. If in equation (14), we make r infinite, we get

$$\frac{1}{f'} = \frac{1}{mf}$$

or

$$mf = f'$$

which answers to the case of a small pencil deviated at a plane surface separating two media of different densities, as air and water. On the supposition that the radiant is in the denser medium, as

in (fig 20), m becomes $\frac{1}{m}$, and this in the preceding equation gives

$$f = m f';$$

that is, to an eye situated without this medium, the distance of the radiant from the deviating surface will appear diminished in the ratio of unity to the index of refraction of the ray in passing from the denser to the other medium. This accounts for the apparent elevation above their true positions of all bodies beneath the surface of fluids, and for the apparent bending of a straight stick when partly immersed in water.

Application to the Deviation of Light by Spherical Reflectors.

41. In reflection, we have only to consider one deviating surface. Equation (14) applies here by making $m = -1$, (17), which reduces it to,

$$\frac{1}{f'} = \frac{2}{r} - \frac{1}{f} \quad . \quad . \quad . \quad . \quad (30)$$

But two cases can arise, and these will be distinguished by the sign of the radius. The reflector may be concave, when r will be positive, or it

may be convex, when r will be negative. Equation (30), and (fig. 21), relate to the first case, which will now be discussed.

If the incident rays be parallel, $\frac{1}{f}=0$, and

$$\frac{1}{f'}=\frac{2}{r}$$

or

$$f'=\frac{r}{2}=\mathrm{F}_{,}$$

Hence the principal focal distance is equal to *half radius*, and equation (30), reduces to

$$\frac{1}{f'}=\frac{1}{\mathrm{F}_{,}}-\frac{1}{f} \quad . \quad . \quad . \quad . \quad (31)$$

As long as $\frac{1}{\mathrm{F}_{,}}>\frac{1}{f}$ or $f>\mathrm{F}_{,}$, f' will be positive, which, since the rays are thrown back from the deviating surface, will correspond to a real focus, and the rays will converge after deviation.

If $\frac{1}{\mathrm{F}_{,}}<\frac{1}{f}$ or $f<\mathrm{F}_{,}$, f' will be negative, and there will be a virtual focus, or the rays will diverge after deviation.

If the radiant be at the centre of curvature, $f=2\,\mathrm{F}_{,}$, and

$$f'=2\,\mathrm{F}_{,}=r$$

or the radiant and focus coincide.

For

$$f > 2F_{,} \text{ or } f > r;$$

$$\frac{1}{f'} > \frac{1}{2F_{,}}, f' < r;$$

or the focus will be between the reflector and centre, and since $\frac{1}{F_{,}} - \frac{1}{f} < \frac{1}{F_{,}}, \frac{1}{f'}, < \frac{1}{F_{,}}$, or $f' > F_{,}$; so that the focus will be found between the centre and principal focus.

For

$$f < 2F_{,} \text{ or } f < r;$$

$$\frac{1}{f'} < \frac{1}{2F_{,}}, f' > r;$$

or the focus will be at a greater distance from the reflector than the centre.

When $f = F_{,}$, $\frac{1}{f'} = 0$, or the focus will be at an infinite distance.

When $f < F_{,}$, f' will be negative, and the rays will diverge after deviation.

If the rays converge before incidence, f will be negative, and equation, (31), becomes

$$\frac{1}{f'} = \frac{1}{F_{,}} + \frac{1}{f}$$

Hence, f' will always be positive, or the rays will converge after deviation.

42. By discussing the several cases that will arise in attributing different signs to r and f, and various values to the latter, we shall find the results in the following

Table.

Reflector.	Incident Pencil.	$\frac{1}{f'}$	Sign of $\frac{1}{f'}$	Reflected Pencil.
Concave $+F_{,}$	Diverging $+f$	$\frac{1}{F_{,}}-\frac{1}{f}$	$f > F_{,}$ $+$	Converges
			$f < F_{,}$ $-$ $f' > f$	Diverges less.
	Converging $-f$	$\frac{1}{F_{,}}+\frac{1}{f}$	$+$ $f' < f$	Converges more.
Convex $-F_{,}$	Diverging $+f$	$-\frac{1}{F_{,}}-\frac{1}{f}$	$-$ $f' < f$	Diverges more.
	Converging $-f$	$-\frac{1}{F_{,}}+\frac{1}{f}$	$f > F_{,}$ $-$	Diverges.
			$f < F'$ $+$ $f' > f$	Converges less.

Hence, in general, concave reflectors tend to collect the rays of light, and convex to disperse them.

43. Differentiating equation (31), we have

$$-\frac{df'}{f'^2} = \pm\frac{df}{f^2};$$

The upper sign corresponds to the case when f and f' have the same signs. Hence the conjugate foci of spherical reflectors will move in *opposite* directions.

44. Equation (30), by making r infinite, reduces to

$$\frac{1}{f'} = -\frac{1}{f}$$

or

$$f' = -f,$$

Which shows, that in all cases of deviation of a pencil by a plane reflector, the divergence or convergence will not be altered; and if the rays diverge before deviation, they will appear after deviation to proceed from a point as far behind the reflector as the actual radiant is in front; but if they converge before deviation, they will be brought to a focus as far in front as the virtual radiant is behind the reflector.

Spherical Aberration.

45. Thus far the discussion has been conducted upon the supposition that the pencil is very small, and that z, the versed-sine of the arc θ, included between the axis and the extreme rays of the pencil, is so small, that all the products of which it is a factor may be neglected. If, however, z be retained

and we find the value of f', in equation (12), by Maclaurin's Theorem, we shall have

$$f' = (f')_0 + \left(\frac{df'}{dz}\right)_0 \cdot z + \left(\frac{d^2 f'}{(dz)^2}\right)_0 \cdot \frac{z^2}{1.2} + \&c.$$

Wherein $(f')_0$ is the value of f', as given by equation (13).

In questions of practice, z is so small that the higher powers than the first may be neglected without impairing greatly the accuracy of the approximation as given by the above series. Retaining, then, only the first power of z, we shall have

$$f' = (f')_0 + \left(\frac{df'}{dz}\right)_0 \cdot z.$$

To find $\frac{df'}{dz}$, we resume equation (12), and differentiate it, regarding f' and z as variable.

$$(f-r)\sqrt{f'^2 - 2r(f'-r)z} = m(f'-r)\sqrt{f^2 - 2r(f-r)z}.$$

$$\frac{(f-r).[f'.df' - rz.d.f' - r(f'-r)dz]}{\sqrt{f'^2 - 2r(f'-r)z}} =$$

$$mdf'\sqrt{f^2 - 2r(f-r)z} - m(f'-r) \cdot \frac{r.(f-r)dz}{\sqrt{f^2 - 2r(f-r)z}}$$

Making $z = 0$, f' assumes the limiting value $(f')_0$

as given by equation (13), and we have,

$$(f-r) . \frac{(f')_0\, d f' - r\,((f')_0 - r)\, d z}{(f')_0} =$$
$$m f . d f' - \frac{m . r . ((f')_0 - r)\,(f-r)}{f} d z.$$

$$(f-r) \left\{ d f' - \frac{r\,((f')_0 - r)}{(f')_0} d z \right\} =$$
$$m f . d f' - \frac{m . r . ((f')_0 - r)\,(f-r)}{f} d z.$$

$$m f . d f' - (f-r)\, d f' = \frac{m . r\,((f')_0 - r)\,(f-r)\, d z}{f}$$
$$- \frac{r((f')_0 - r)\,(f-r) . d z}{(f')_0}$$

$$\frac{d f'}{d z} = \frac{r . (f-r)\,((f')_0 - r)}{\overline{m-1} . f + r} . \left(\frac{m}{f} - \frac{1}{(f')_0}\right).$$

but from equation (13), we have

$$(f')_0 - r = \frac{m r f}{\overline{m-1} . f + r} - r = \frac{r\,(f-r)}{\overline{m-1} f + r}$$

hence,

$$\frac{d f'}{d z} = \overline{(f')_0 - r}^2 . \left(\frac{m}{f} - \frac{1}{(f')_0}\right) . \quad . \quad . \quad . \quad . \qquad (32)$$

and

$$f' = (f')_0 + \overline{(f')_0 - r}^2 . \left(\frac{m}{f} - \frac{1}{(f')_0}\right) . z \quad . \quad . \quad . \quad . \qquad (33)$$

46. This equation relates to a single surface; if

we pass to a second, and limit the investigation to the case where $m = \frac{1}{m'}$, which supposes the light to pass to the same medium from which it entered the first deviating surface, the result will appertain to a single lens.

In equation (15), we have f'' a function of f'; the latter, in equation (12), is a function of z; and calling z', the versed-line of the arc θ' at the second deviating surface, included between the axis and the extreme ray deviated at the first surface, f'' will also obviously depend upon z', which, together with z, was neglected in finding equation (19). Hence we shall have f'', a function of (z, z'). By the general development of a function of two variables, limiting the series to the terms involving the first powers of the variables, for the reason given in art. 45.

$$f'' = (f'')_0 + \left\{ \left(\frac{df''}{dz}\right)_0 z + \left(\frac{df''}{dz'}\right)_0 z' \right\} \quad . \quad . \quad . \quad . \quad (34).$$

To obtain the first partial differential co-efficient, we have

$$\left(\frac{df''}{dz}\right)_0 = \frac{df''}{df'} \times \frac{df'}{dz},$$

and to obtain $\frac{df''}{df'}$, it will be sufficient to take equa-

tion (15). Neglecting t, and making $m' = \frac{1}{m}$ we find,

$$\frac{df''}{df'} = \frac{m(f'')_0^2}{(f')_0^2},$$

and $\frac{df}{dz}$, is given by equation (32).

The second partial differential co-efficient, is found at once from equation (32), by changing f into f', r into r', z into z', and m into $\frac{1}{m,}$ which will give,

$$\left(\frac{df''}{dz'}\right)_0 = ((f'')_0 - r')^2 \left\{ \frac{1}{mf'} - \frac{1}{(f'')_0} \right\}.$$

These several values being substituted in equation (34), give

$$\begin{aligned} f'' = (f'')_0 + \frac{m(f'')_0^2}{(f')_0^2} \cdot \overline{((f')_0 - r)}^2 \cdot \left(\frac{m}{f} - \frac{1}{(f')_0}\right) . z \\ + \overline{(f'')_0 - r'}^2 \cdot \left(\frac{1}{m(f')_0} - \frac{1}{(f'')_0}\right) z' \quad . \quad (35) \end{aligned}$$

If the arcs θ and θ' be taken so small that z and z', their versed-sines, may be neglected, this equation reduces to

$$f'' = (f'')_0$$

as in equation (19). Subtracting this last equation from (35), we have

$$f'' - (f'')_0 = \frac{m\,(f'')_0{}^2}{(f')_0{}^2} \cdot \overline{(f')_0 - r}^{\,2} \left(\frac{m}{f} - \frac{1}{(f')_0}\right) \cdot z$$

$$+ \overline{(f'')_0 - r'}^{\,2} \cdot \left(\frac{1}{m(f')_0} - \frac{1}{(f'')_0}\right) z' \quad . \quad (36)$$

Now f'', is the focal distance of the rays which are incident at the distances θ, θ', from the common axis of the deviating surfaces, and $(f'')_0$, the focal distance of those incident very near; hence the difference, or $f'' - (f'')_0$, will be the length of that portion of the axis upon which will be situated the foci of the rays between the boundary of the pencil and its central ray, supposed coincident with the axis. This wandering of the deviated rays from a single focus, thus shown to exist in the case of spherical surfaces, is called *spherical aberration.*

Let O, be the focus corresponding to $(f'')_0$, in figure (22); O′, the focus corresponding to f''. The distance O O′ is called the *longitudinal aberration;* and if a perpendicular to the axis be drawn through O and produced till it meet the deviated ray through O′ in *b*, O *b*, is called the *lateral aberration.* Calling the longitudinal aberration *a*, we shall have,

$$a = \frac{m\,(f'')_0{}^2}{(f')_0{}^2} \cdot ((f')_0 - r)^2 \cdot \left(\frac{m}{f} - \frac{1}{(f')_0}\right) \cdot z$$

$$+ ((f'')_0 - r')^2 \cdot \left(\frac{1}{m(f')_0} - \frac{1}{(f'')_0}\right) z' . \quad . \quad (37)$$

C, being the centre of curvature of the second deviating surface M M′, $D P = r \sin \theta$, is called the *radius of the aperture*; and if z and z' be equal, or nearly so, it follows from equation (37), that *the longitudinal aberration for a given value of f, in a given lens, will vary as the square of the radius of the aperture.*

47. Calling the lateral aberration b, the longitudinal a, as before, the similar triangles of the figure give

$$\frac{b}{a} = \frac{r \sin \theta}{f'' + z},$$

but z may be neglected as compared with f'', within ordinary limits, hence

$$b = \frac{a \,.\, r \,.\, \sin \theta}{f''} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

and as a varies as $(r \sin \theta)^2$, *the lateral aberration will vary as the cube of the radius of the aperture.*

48. Resuming equation (33), and making $m = -1$, it reduces, calling a the longitudinal aberration, to

$$a = f' - (f')_o = -\left[((f')_o - r)^2 . \left(\frac{1}{f} - \frac{1}{(f')_o}\right)\right] z \quad . \quad (39)$$

In figure (23), O′ and O, being the foci corresponding to the values f' and $(f')_0$, we shall have by inspection, denoting the lateral aberration by b,

$$\frac{b}{a} = \frac{r \sin \theta}{f' - z},$$

or

$$b = \frac{a \, . \, r \sin \theta}{f'}$$

from this, and equation (39), we infer, that in reflection at spherical surfaces as in refraction, *the lateral aberration varies as the cube, and the longitudinal as the square of the radius of the aperture.*

49. Since the rays deviated at spherical surfaces are not brought to a single focus, it becomes a matter of some interest to ascertain the magnitude and position of the least space that will contain them all : this space is called the *circle of least diffusion.*

Let O (fig. 24), be the focus corresponding to the value $(f'')_0$, and O′ to the value f''. If a ray be deviated on the side of the axis opposite to θ, and at a distance θ'', less than θ, it will meet the axis at O″, between O and O′, and intersect the extreme ray passing through O′ produced at y. Another ray deviated at a distance θ''', greater than θ'', will belong to a focus between O′ and O″,

but may meet the extreme ray produced to the right or left of y, making the length of $x\,y$ greater or less than that corresponding to θ''; hence if a value for $x\,y$ be found answering to a maximum, all the rays will necessarily pass through the circular space having this value for radius.

Call a the longitudinal aberration for θ; a'' that for θ''; b and b'' the corresponding lateral aberrations; x, the distance of the circle of least diffusion from O′, and y its radius. Then y will be a maximum when x is so.

Equation (38), gives

$$\frac{b}{a} = \frac{r \sin \theta}{f''},$$

$$\frac{b''}{a''} = \frac{r \sin \theta''}{f''},$$

by division,

$$\frac{b\,a''}{b''\,a} = \frac{\sin \theta}{\sin \theta''},$$

$$\frac{x}{y} = \frac{a}{b},$$

$$\frac{a - a'' - x}{y} = \frac{a''}{b''},$$

by division,

$$\frac{a - a'' - x}{x} = \frac{a''\,b}{b''\,a} = \frac{\sin \theta}{\sin \theta''},$$

hence,

$$a - a'' = x . \frac{\sin\theta + \sin\theta''}{\sin\theta''};$$

but from the relation between the longitudinal aberration and radius of aperture, we have

$$\frac{a''}{a} = \frac{\sin^2\theta''}{\sin^2\theta},$$

or

$$a - a'' = a . \frac{\sin^2\theta - \sin^2\theta''}{\sin^2\theta} = x \frac{\sin\theta + \sin\theta''}{\sin . \theta''},$$

hence,

$$x = a \frac{\sin\theta'' (\sin\theta - \sin\theta'')}{\sin^2\theta}.$$

$$\frac{d x}{d \theta''} = \frac{\sin\theta . \cos\theta'' - 2 \sin\theta'' . \cos\theta''}{\sin^2\theta} = 0$$

or

$$\sin\theta'' = \tfrac{1}{2} \sin\theta,$$

hence,

$$x = \tfrac{1}{4} a;$$

and

$$y = \tfrac{1}{4} b.$$

Again,

$$\frac{d^2 x}{d \theta^{2\,''}} = -\frac{\sin\theta'' + 2}{\sin\theta};$$

y is, therefore, a maximum.

50. In equation (33), which answers to one surface, there will be no aberration when either of the factors in the last term of the second member is equal to zero. Taking the first factor,

$$(f')_0 - r = 0,$$

we obtain from equation (13),

$$f = r = (f')_0 :$$

and the second factor,

$$\frac{m}{f} - \frac{1}{(f')_0} = 0,$$

replacing $\frac{1}{(f')_0}$, by its value given in equation (14), and reducing, we have

$$f = r\,(m + 1)$$

If r be positive, the deviating surface will have its concavity turned towards incident light; if negative, its convexity. In the first case, the incident rays will diverge, and if the first factor be zero, they will diverge from the centre of curvature and will, of course, undergo no deviation; but if the second factor reduce to zero, the rays will diverge from a point beyond the centre and distant from it equal to $m\,r$. In the second case, viz., where r is

negative, the incident rays will converge to the centre of curvature when the first factor is zero, or to a point beyond the centre, when the second factor is zero.

In the case of reflection, $m = -1$, and the second factor cannot reduce to zero.

Generally, then, every spherical refracting surface has two points on its axis, so related that all rays proceeding from or to one of them, will after deviation, proceed from or to the other. These points have been called *aplanatic foci;* the first being called the aplanatic focus for incident, the other for refracted rays. The distance of the first from the surface is given by

$$f = (m + 1) . r$$

and that of the other by

$$f' = \frac{m+1}{m} . r$$

51. A similar analysis might be made of equation (37), but it would lead to investigations too long and difficult for an elementary work like this. The total amount of longitudinal aberration due to the action of two surfaces may be readily found, however, from this equation.

Example 1*st.* Required the longitudinal aberra-

tion due to the action of a glass concavo-convex lens on parallel or solar rays, wherein,

$$m = \tfrac{3}{2},$$
$$r = 1,$$
$$r' = \tfrac{5}{3},$$

In equations (14) and (19), $\frac{1}{f}$, being $= 0$,

$$f' = 3$$
$$f'' = 5$$

hence from equation (37),

$$a = \left\{ -\frac{3}{2} \cdot \frac{5.^2}{3.^2} \cdot 2^2 \cdot \frac{1}{3} + \frac{10.^2}{3.^2} \left(\frac{2}{3} \cdot \frac{1}{3} - \frac{1}{5} \right) \right\} \cdot z = \frac{430}{81} z$$

z and z' being supposed equal, and if θ be taken 2°.

$$a = -\cdot 003$$

Example 2d. Required the longitudinal aberration for parallel rays in a double concave lens, wherein,

$$r = r' = 5$$
$$m = \tfrac{3}{2}$$
$$z = .0006$$

$\frac{1}{f}$, being zero as before, and r' being negative, equations (14) and (19), give

$$f' = 15$$
$$f'' = 5$$

and equation (37),

$$a = -.023.$$

Example 3d. Required the aberration for parallel rays in the case of a double convex lens, having as before,

$$r = r' = 5$$
$$m = \tfrac{3}{2}$$
$$z = .0006$$

r, being negative, and r' positive,

$$f' = -15$$
$$f'' = -5$$

and

$$a = .01$$

In each of these examples, the aberration being of a sign contrary to that of f'', as given by equation (19), its tendency is to shorten the focal distance. And this is true for all single spherical lenses constructed of any known medium, no matter what

the degree of curvature. The destruction of spherical aberration in a single lens, for parallel rays, is, therefore, impossible, though by the use of two lenses placed close together, it may be effected in a variety of ways. Such combinations are said to be *aplanatic*.

Oblique Pencils.

52. Heretofore the radiant has been taken on the common axis of the deviating surfaces, and the axis of the pencil supposed to coincide with the same line; the axis or central ray of the pencil being, in this position, normal to all the surfaces, has either undergone no deviation in passing from one medium to another, or been driven back upon itself when not permitted to enter the media before it; the other rays of the pencil have, moreover, been deviated so as to intersect the axis when not rendered parallel to it, and consequently to have their foci upon that line.

But when the radiant is taken off the common axis, the rays of the pencil, including its central one, will in general be oblique to the surfaces, and a new state of things will arise. The pencil is said to be *oblique*, and it is now proposed to investigate the circumstances attending its deviation.

Oblique Pencil through the Optical Centre.

53. We have seen, article (20), that a ray undergoes no deviation when it passes through a medium bounded by two parallel planes. If, then, in the new position of the radiant, we suppose the pencil to increase indefinitely, there may always be found one ray that will enter and leave the lens at points, where tangent planes to its two surfaces are parallel. This ray being taken as the axis of a very small pencil proceeding from the assumed radiant, will contain the focus of the others, the distance ofwhich from the lens, in very moderate obliquities, will be measured by f'', given in equation (19).

To find where the ray referred to, after undergoing one deviation, intersects the axis of the surfaces, let figure 25 represent a section of a concavo-convex, and figure 26 that of a double convex lens. Call the distance between the surfaces, measured on the common axis, t, and let e be the distance from the first surface to where the line, joining the points at which any two tangents are parallel, cuts the axis. Then since the radii at these points are parallel, the similar triangles of figure 25 give,

$$\frac{r}{r-e} = \frac{r'}{r'-t-e},$$

hence,

$$e = \frac{rt}{r'-r};$$

and of figure 26,

$$\frac{r}{r-e} = \frac{r'}{r'-t+e}$$

hence,

$$e = \frac{r\,t}{r'+r};$$

or generally

$$e = \frac{r\,t}{r' \mp r}; \quad . \quad . \quad . \quad (40)$$

the upper sign answering to the case where the radii have the same sign.

This value of e, being constant for the same lens, it follows that all rays which emerge from a lens parallel to their original directions, after deviation at the first surface, proceed in directions having a common point on the axis. This point is called the *optical centre of the lens*, and may lie between the surfaces or not, depending upon the form of the lens.

If we suppose but one surface, that there may be no deviation, e must be equal to r; and the axis of the pencil must pass through the centre of curvature.

Oblique Pencil not passing through the Centre.

54. Now let the radiant be assumed at pleasure, either on or off the axis, and a ray not passing

through the optical centre be taken as the axis of a small pencil ; this ray will, of course, undergo deviation as well as all others of the pencil, and the circumstances attending the directions of the deviated rays will be different from those before considered.

Let O (fig. 27), be the position of the radiant, O D, O D′, two contiguous rays in a plane passing through the radiant and the centre of curvature C, and O′ D, O′ D′, the directions of these rays after deviation. Putting O D $= u$; O D′ $= u + d\,u$; D D′ $= d\,s$; D O′ $= u'$; D′ O′ $= u' + d\,u'$, and C D $=$ C D′ $= r$; we shall have

$$d\,u = d\,s\,.\sin\varphi,$$

$$d\,u' = d\,s\,.\sin\varphi',$$

whence,

$$\frac{d\,u}{d\,u'} = \frac{\sin\varphi}{\sin\varphi'} = m$$

or

$$d\,u - m\,d\,u' = 0 \quad . \quad . \quad . \quad . \quad (40)$$

Joining the points O and O' with C, and calling O C, k, and O′ C, k', the triangles D O C and C O′ D will give,

$$k^2 = u^2 + r^2 - 2\,r\,u\,.\cos\varphi$$

$$k'^2 = u'^2 + r^2 - 2\,r\,u'\cos\varphi';$$

and since the position of O′, is determined by the intersection of two contiguous rays, k' will remain constant in passing from D to D′; hence,

$$0 = u \, . \, d\,u - r \, . \, d\,u \, . \cos\varphi + r\,u \, . \sin\varphi \, . \, d\,\varphi$$

$$0 = u' \, . \, d\,u' - r \, . \, d\,u' \, . \cos\varphi' + r\,u' \sin\varphi' \, . \, d\,\varphi'$$

or

$$u = -\frac{u - r\cos\varphi}{r \, . \sin\varphi} \, . \, \frac{d\,u}{d\,\varphi}$$

$$u' = -\frac{u' - r\cos\varphi'}{r \, . \sin\varphi'} \, . \, \frac{d\,u'}{d\,\varphi'}$$

dividing the second by the first,

$$\frac{u'}{u} = \frac{u' - r\cos\varphi'}{u - r\cos\varphi} \, . \, \frac{\sin\varphi}{\sin\varphi'} \, . \, \frac{m \, . \, d\,u'}{d\,u} \, . \, \frac{d\,\varphi}{m \, . \, d\,\varphi'} \, .$$

replacing $\frac{m \, . \, d\,u'}{d\,u}$, by its value in equation (40),

and $\frac{d\,\varphi}{m \, . \, d\,\varphi'}$, deduced from equation (1), we get,

$$\frac{u'}{u} = \frac{u' - r\cos\varphi'}{u - r\cos\varphi} \, . \, \frac{\sin\varphi}{\sin\varphi'} \, . \, \frac{\cos\varphi'}{\cos\varphi} \, .$$

whence

$$u' = \frac{u\,r \, . \cos\varphi' \tan . \, \varphi}{u . \tan\varphi - (u - r\cos\varphi)\tan\varphi'} \quad . \quad . \qquad (42)$$

The position of the radiant being given, this equation will determine that of the corresponding conjugate focus for those rays of an indefinitely small pencil which are contained in a plane passing through the radiant and centre of curvature. If that plane containing the axis of the pencil, and which is called the *principal plane*, be revolved about the line O C, it will cut in succession different sets of rays from the pencil, whose foci will also be determined by equation (42), and all of which will intersect O C. These foci will evidently lie in a small curve described by the point O′, in its motion about O C; and the plane of this curve, called the *secondary plane*, will be perpendicular to the principal plane. Hence, considering the small arc as a right line, we may infer *that in any small oblique pencil, all the rays, after deviation, will pass through two lines in planes at right angles to each other.* These are called *focal lines*, and their property of intersecting all the deviated rays, *astigmatism.*

55. To ascertain the form which this small deviated pencil takes, let the transverse section of the pencil, before deviation, be a circle whose diameter is λ. An oblique section by the deviating surface will be an ellipse whose axes are λ, and λ sec. φ, and the deviated pencil will assume a conoidal shape, the sections of which, by the principal plane and a secondary plane containing the axis of the conoid, will be triangular, having their ver-

tices in the focal lines, and their bases in the deviating surface. The transversal section will in general be an ellipse. Suppose one of the latter sections to be made at a distance x, (fig. 28), from the deviating surface and parallel to it; and let v, be the length of the axis D B, between the same surface and the focal line in the principal plane; then calling l, the diameter of the transversal section in this latter plane, and h that in the secondary plane, we have,

$$u' : u' - x :: \lambda \sec \varphi : l$$

$$v : v - x :: \lambda : h$$

or

$$l = \frac{u' - x}{u'} . \lambda . \sec \varphi . \quad . \quad . \quad (43)$$

$$h = \frac{v - x}{v} . \lambda . \quad . \quad . \quad . \quad . \quad (44)$$

as x increases, l and h will decrease.

When

$$x = u',$$

$$l = 0$$

$$h = \frac{v - u'}{v} \lambda.$$

or the ellipse becomes the *secondary focal line.*

If x be greater than u', l will increase, and h will still decrease as long as $x < v$; l and h will, therefore, be equal at some point where the section will become a circle. To find this point we have,

$$\frac{u'-x}{u'} \cdot \lambda \cdot \sec \varphi = \frac{v-x}{v} \cdot \lambda$$

or

$$x = \frac{u'(1+\cos\varphi)}{1+\frac{u'}{v} \cdot \cos\varphi'};$$

which will give the position of the circle; and its diameter will be given by putting this value for x, in equation (43) or (44).

Making this substitution and reducing, we get

$$h = \lambda \cdot \frac{v-u'}{v+u'\cos\varphi}.$$

The circle of which this is the diameter, is called the *circle of least confusion*, because within it the rays approach most nearly to convergence.

If x continue to increase and become equal to v,

$$h = 0$$

$$l = \lambda \cdot \sec\varphi' \cdot \frac{u'-v}{u'},$$

and the ellipse becomes a line equal to the primary

focal line, the length of which is, therefore, known. Beyond this, both l and h increase indefinitely.

To pursue the oblique pencil through a second deviating surface, would lead to difficulties which we will avoid, our principal object having been attained, viz : to show the want of accurate convergence in any system of rays which includes pencils of considerable obliquity, and to prepare us for understanding the action of deviating surfaces in the formation of optical images.

56. Equation (42), may be put under the form

$$u' = \frac{u\,r\,.\cos\varphi'\,.\,\dfrac{\tan\varphi}{\tan\varphi'}}{u\,.\dfrac{\tan\varphi}{\tan\varphi'} - (u - r\cos\varphi)} \quad . \quad . \quad . \quad (45)$$

The radiant being supposed on the axis, or $\varphi = 0$; we have $u' = f'$; $u = f$, and

$$\frac{\sin\varphi}{\sin\varphi'} = \frac{\tan\varphi}{\tan\varphi'} = m$$

hence equation (45), reduces to

$$f' = \frac{r\,m\,f}{(m-1)f + r},$$

the same as equation (13).

If the pencil be deviated by reflection, we shall have

$$\varphi = \varphi'$$

$$\frac{\tan \varphi}{\tan \varphi'} = m = -1$$

Substituting in (45), and taking the reciprocals and reducing, we get

$$\frac{1}{u'} + \frac{1}{u} = \frac{2}{r \cos \varphi}$$

Taking the radiant on the axis, u' will become f', and u, f; and this last equation reduces to

$$\frac{1}{f'} + \frac{1}{f} = \frac{2}{r}$$

same as equation (30).

Optical Images.

57. The surface of every luminous body is made up of a vast number of radiants, from each of which, rays of light are supposed to proceed in all directions not intercepted by the body itself. These rays, of course, cross each other; and if any deviating surface be presented, it becomes the common base of a multitude of pencils, whose vertices are

the radiants constituting the surface of the body. If the body be so situated with respect to the deviating surface, that each pencil shall have one of its rays passing through the optical centre, it is obvious from article (53), that those rays of each pencil in the immediate vicinity of this one, assumed as an axis, will, in the case of moderate obliquities, be brought to a focus after deviation; and hence, for each radiant, there will be a corresponding conjugate. These conjugate foci make up a second luminous surface, from which rays will proceed as from the original body; and this surface is called the *image of the body*, because to an eye so situated as to receive the rays as they diverge from it, the object, though often modified in shape and size, will seem to occupy the position of the new surface.

The formation of an optical image consists, therefore, in so deviating the rays of light which proceed from any object, that the whole or a part of them shall unite and again proceed from some new position, in a manner similar to that of their leaving the object. In general, but a part of the rays will be deviated to satisfy these conditions, for those remote from the axis of each pencil cannot, in consequence of aberration and astigmatism, be brought to accurate convergence.

58. To ascertain the relation, between an object and its image, let us suppose the deviation to be produced by a lens, so thin that its thickness may be neglected. The optical centre, in this

case, may be taken as the origin of co-ordinates. Denoting by l, the distance from this point to any assumed point in the object, and writing this quantity for f, in equation (23), we get

$$f'' = \frac{F_{\prime\prime}}{1 + \frac{F_{\prime\prime}}{l}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Let the object (fig. 29), be a right line, perpendicular to the axis of the lens. Call θ, the angle included between the axis of any oblique pencil and the axis of the lens. When the pencil becomes direct, θ will be zero, and l will equal f. But, generally, we have

$$l = \frac{f}{\cos\theta};$$

this in equation (46), reduces it to

$$f'' = \frac{F_{\prime\prime}}{1 + \frac{F_{\prime\prime}}{f}\cos\theta} \quad . \quad . \quad . \quad (47)$$

which is the polar equation of a conic section. Comparing it with

$$r = \frac{A\,(1 - e^2)}{1 + e\cos v},$$

(See Davies' An. Geom., Book IV. Sch. 5th), we get

$$f'' = r,$$

$$F_{\prime\prime} = A\,(1 - e^2) = \frac{B^2}{A}, \quad . \quad . \quad (48).$$

$$e = \frac{F_{\prime\prime}}{f}, \quad . \quad . \quad (49).$$

$$\theta = v.$$

For the same lens, $F_{\prime\prime}$ is constant; its value $\frac{B}{A}$, in equation (48), which is the radius of curvature at the vertex, is also constant.

From equation (49), it is easily seen that the curve will be the arc of a circle, ellipse, parabola, hyperbola, or a right line, one of the varieties of the hyberbola, according as

$$\frac{F_{\prime\prime}}{f} = 0,$$

$$\frac{F_{\prime\prime}}{f} < 1,$$

$$\frac{F_{\prime\prime}}{f} = 1,$$

$$\frac{F_{\prime\prime}}{f} > 1,$$

$$\frac{F_{\prime\prime}}{f} = \infty.$$

or according as the distance of the object is infinite; greater than the principal focal distance of the lens; equal to this distance; less than this distance; or zero.

If, now, the section represented in figure 29 be supposed to revolve about the axis of the lens, the object will generate a plane, and the image a curved surface whose nature will depend upon the distance of the object.

We have seen, article (34), that a positive value for f'', answers to an imaginary, and a negative value, to a real focus; so, if the points of the image be indicated by positive values for f'', the image will be imaginary; if by negative values, real. $F_{,,}$ for a concave lens is positive, and equation (47), answers to this case. $F_{,,}$ for a convex lens is negative, and equation (47) becomes

$$f'' = -\frac{F_{,,}}{1 - \frac{F_{,,}}{f}\cos\theta}$$

and the image will always be real as long as

$$\frac{F_{,,}}{f}\cos\theta < 1$$

or

$$\frac{f}{\cos\theta} > F_{,,}$$

That is, if from the optical centre (fig.30), with a radius equal to the principal focal distance, we describe the arc of a circle, and this arc cut the object, all that part of the object included between the points of intersection A and A′, will have no image, or the image will be virtual, while the parts without these limits will have real images; if the distance of the object exceed that of the principal focus, the whole image will be real.

In general then, in a concave lens, the image is always imaginary or virtual, and in a convex lens, real, as long as the distance of the object is greater than the principal focal distance; within that limit it is also imaginary.

59. If the linear dimensions of the object be small, as compared with its distance from the optical centre, cos θ, in equation (47), may be taken equal to unity; making this supposition and reducing, we get for a convex lens

$$f'' = -\frac{F_{,,} f}{f - F_{,,}} = -\frac{F_{,,}}{1 - \frac{F_{,,}}{f}} \quad . \quad . \quad (50)$$

this value of f'', being constant for the same position of the object, the image will be a circle; and since the axes of all the pencils intersect at the optical centre, we may, without material error, assume every object, in the case supposed, a small arc of a circle having the same centre as the circu-

lar image. The object and image will, therefore, be similar, and if any linear dimension of the former be represented by δ, and the corresponding dimension of the latter by δ', we shall have (fig. 31),

$$\frac{\delta'}{\delta} = \frac{f''}{f};$$

substituting this value in equation (50), it reduces to

$$\frac{\delta'}{\delta} = -\frac{F_{\prime\prime}}{f - F_{\prime\prime}} \quad . \quad . \quad . \quad (51)$$

If the distance of the object from the lens be equal to twice the principal focal distance,

$$\frac{\delta'}{\delta} = -1,$$

or the object and image will be of the same size.

If the distance of the object exceed double the principal focal distance, the image will be less than the object; if its distance be less than double the principal focal distance, the image will be greater than the object.

It also follows from the fact just alluded to, viz: the intersection of the axes of the several pencils at the optical centre, that the *real image will always be inverted with respect to the object*. If the rays which proceed from this image be again deviated so as to form a second image, it will be in-

verted with respect to the first image, but erect as regards the object.

60. If an image be formed by deviation at a single surface, its points will (fig. 32), be readily found by means of equation (26). The optical centre, in this case, being at the centre of curvature.

Writing f for c, and f' for c', that equation becomes

$$\frac{1}{f'} = \frac{m-1}{r} + \frac{m}{f}.$$

making $f = \infty$

$$\frac{1}{f'} = \frac{m-1}{r} = \frac{1}{F_{,}} \quad . \quad . \quad . \quad (k)$$

hence,

$$\frac{1}{f'} = \frac{1}{F_{,}} + \frac{m}{f};$$

or

$$f' = \frac{f F_{,}}{f + m F_{,}} . = \frac{F_{,}}{1 + \frac{m F_{,}}{f}}.$$

For an oblique pencil passing through the optical centre, we have, on the supposition that the

object is a right line perpendicular to the axis of the surface,

$$f' = \frac{F_{\prime}}{1 + \frac{m F_{\prime}}{f} \cos \theta.}$$

wherein $\frac{\cos \theta}{f} = l$, as in art. (58), and if the image be formed by reflection, $m = -1$; hence

$$f' = -\frac{F_{\prime}}{1 + \frac{F_{\prime}}{f} \cos \theta} \quad . \quad . \quad . \quad (52)$$

since $F_{\prime}$ becomes negative, equation (k). This is a polar equation of a conic section, the nature of which will result from the relation of $F_{\prime}$ to f. It will be an ellipse, parabola, or hyperbola, according as

$$f > F_{\prime}; \; f = F_{\prime}; \text{ or } f < F_{\prime}.$$

The axis of the pencils crossing at the centre C, (fig. 33), the image when real will be inverted as respects the object, and when the object is small, we shall have from the similar triangles of the figure, *the linear dimensions of the object to those of the image as the distance of the object from the centre, to that of the image from the same point.*

We get the point in which the image cuts the axis by making

$$\theta = 0,$$

or

$$f' = -\frac{F_{,}}{1+\frac{F_{,}}{f}} = -\frac{f}{1+\frac{f}{F_{,}}} \quad . \quad . \quad (53)$$

This value of f' being negative, the image will be found on the left of the centre, the distance f having been taken positive to the right. As long as f is positive, the image will lie between the centre and reflector; f' will be less than f, and the image, consequently, less than the object. When f is zero, f' will also equal zero, and the object and image will be equal and occupy the centre. When f becomes negative, or the object passes between the centre and reflector, f' will be positive as long as $f < F_{,}$, and the image will pass without; f' will be greater than f, or the image will be greater than the object. When f, being still negative, is equal to $F_{,}$, or the object is in the principal focus, the image will be infinitely distant. The object still approaching the reflector, f' will be greater than $F_{,}$; f' becomes negative again and the image will approach the reflector from behind it, and will be greater than the object till $f = 2F_{,}$ or the object be in contact with the reflector, when f' will equal f, and the image and object be of the same size. If now, the object

pass the deviating surface, and reflection take place at its other face, we shall have the case of a convex reflector, and equation (53) becomes,

$$f' = -\frac{F_{,}}{1 - \frac{F_{,}}{f}} = \frac{f}{1 - \frac{f}{F_{,}}} \quad . \quad . \quad . \quad (54)$$

This value of f', is always negative, greater than $-F$, and less than $-2F_{,}$, for all values of f, between $-2F_{,}$ and infinity, or for any position of the object from the surface of the reflector to a point infinitely distant in front. In the latter position, f' is equal to $-F_{,}$, or the image is in the principal focus. It follows also, that the image, which will always be virtual, will be elliptical, erect, and smaller than the object.

Caustics.

61. In discussing the deviation of oblique pencils at spherical surfaces, we found in equation (42), a general value for the length of any deviated ray in the principal plane, included between the deviating surface and the point in which it is intersected by the ray next in order. This value depends upon u, φ, and r, or the position of the radiant, the angle of incidence, and radius of the surface; and for any assumed radiant, it is obvious from the na-

ture of the function, that the points of intersection of the consecutive rays will not all have the same position, but will trace out a curve to which the deviated rays will be tangent. This curve is called a *caustic*.

To illustrate, let us suppose that the radiant is infinitely distant, or the rays parallel; equation (42) will reduce to

$$u' = \frac{r \cos \varphi' \,.\, \tan \varphi}{\tan \varphi - \tan \varphi'} \quad . \quad . \quad . \quad . \quad (55)$$

or

$$u' = \frac{r \cos^2 \varphi' \,.\, \sin \varphi}{\sin (\varphi - \varphi')} \quad . \quad . \quad . \quad . \quad (56).$$

and to construct this, let A B (fig. 34), be an incident, and B O, the corresponding deviated ray; C, the centre of curvature. Draw C n perpendicular to B O, $n\,s$ perpendicular to the radius B C, and s O parallel to A B. O, will be a point in the caustic: for

$$\text{B}\,n = r \cos \varphi'$$

$$\text{B}\,s = \text{B}\,n \cos \varphi' = r \cos^2 \varphi'$$

and in the triangle B s O, we have

$$\text{B O} = \frac{r \cos^2 \varphi' \,.\, \sin \varphi}{\sin (\varphi - \varphi')} = u'.$$

If we suppose $\varphi = 90°$, or the incident ray tangent to the deviating surface, equation (56), reduces to

$$u' = r \cos \varphi',$$

draw C O′ perpendicular to the deviated ray B′ O′, and O′, will be the corresponding point in the caustic, and is evidently one of the limits of the curve.

To find the other limit which is on the axis A″ C, and which answers to $\varphi = 0$, we cannot use equation (56), since it assumes the form of indetermination ; equation (55) will, however, give the required point. Putting it under the form,

$$u' = \frac{r \cos \varphi'}{1 - \frac{\tan \varphi'}{\tan \varphi}}.$$

and recollecting that in the case supposed,

$$\frac{\tan \varphi'}{\tan \varphi} = \frac{\sin \varphi'}{\sin \varphi} = \frac{1}{m}$$

it reduces to,

$$u' = \frac{m\, r}{m - 1}.$$

and taking B″ O″, equal to this value of u', O″ O O′, will be that branch of the caustic resulting from deviation between the limits B′ and B″. The same thing will, of course, take place on the opposite side of the axis, and the caustic will have two branches uniting on the axis at O″.

If in equation (42),

$$u \tan \varphi - u \tan \varphi' + r \cos \varphi \tan \varphi' = 0$$

or

$$u = -\frac{r \cos \varphi . \tan \varphi'}{\tan \varphi - \tan \varphi'} = -r \cos^2 \varphi . \frac{\sin \varphi'}{\sin (\varphi - \varphi')}.$$

u' will be infinite, and the caustic will be divided into two branches, lying on opposite sides of the deviating surface; for, any two values of φ, the one greater, and the other less than that which will satisfy this equation, will give u' opposite signs. One of these branches will, of course, be virtual, and the deviated ray answering to φ, as given by the above equation when u is assumed, will be an *asymptote* to both branches.

62. Taking, in the general equation (42), $m = -1$, or $\varphi = -\varphi$, the resulting formula will belong to the

caustic produced by reflection. Making the reduction, we get,

$$u' = \frac{u\,r\cos\varphi}{2\,u - r\cos\varphi} \quad . \quad . \quad . \quad (57)$$

or

$$u' = \frac{r \,.\, \cos\varphi}{2 - \frac{r\cos\varphi}{u}.} \quad . \quad . \quad . \quad (58)$$

The rays being supposed parallel, we have

$$u' = \tfrac{1}{2}\,r\cos\varphi.$$

the construction of which is very easy. Let A B (fig. 35), be any incident, and B n the corresponding reflected ray; draw C n perpendicular to the latter, and take B O equal to one half B n, or one fourth the chord B S; O will be a point in the caustic. Making $\varphi = 90°$, u will equal zero; making $\varphi = 0$, u' will equal one half radius, and the caustic will commence at B′, and terminate in the principal focus.

If deviation take place at the convex surface of the reflector, r will be negative, and

$$u' = -\tfrac{1}{2}\,r\cos\varphi;$$

the caustic will be virtual and similar in all respects to that above.

With C as a centre, and a radius equal to one fourth the diameter of the reflector, describe the circle O″ V; with t, the middle of V B, as a centre and radius Vt, describe the circle V O B, which will pass through O; the arc V O″ will be equal to the arc V O, and the caustic will be an epicycloid described by the motion of the circle V O B on O″ V. For, join V O and t O; B O being equal to half B n, and B V half B C, V O and C n are parallel; V O B is a right angle, and O is in the circumference of which V B is the diameter. The angle C B O = A B C = B C B″ = $\frac{1}{2}$ V t O; but C V = 2 t V, hence V O″ = V O.

If the radiant be taken at the extremity of the diameter, u (fig. 36), will be equal to $2\ r \cos \varphi$, and this substituted in equation (57), will reduce it to,

$$u' = \tfrac{2}{3}\, r \cos \varphi = \tfrac{1}{3}\, u = \tfrac{2}{3}\, \mathrm{B}\, n.$$

C n, being as before perpendicular to B S.

With the centre C, and radius equal to one third the radius of the reflector, describe the circle O″V; with the centre t, the middle of V B, and radius t V, describe the circle V O B, which will pass through O; the arc V O″ will equal the arc V O, and the caustic will again be an epicycloid generated by the motion of the circle V O B upon O″ V. For, B O is two thirds of B n, and B V two

thirds of B C, and the angle V O B, consequently, a right angle; the angle B″ C B = 2 C B A = 2 C B S = O t V, and C V and V t being equal, the arcs O″ V and V O are equal.

When the radiant passes within the circle, and reflection takes place in all directions, there will be some position for the incident ray, giving $u = \frac{1}{2} r \cos \varphi$, which will render the denominator in equation (57), equal to zero. The caustic in this case will be divided into two branches with cusps at O″, O′ and O‴ (fig. 37), and infinite branches extending along the reflected ray answering to that value of φ, whose cosine is equal to $\frac{2u}{r}$; this ray will be an asymptote to both branches.

63. The subject of caustics being one of curiosity rather than utility, it is perhaps unnecessary to our present purpose to pursue it further; it will, therefore, be terminated by a general method for finding the equation of the caustic curve from that of the deviating surface, when the position of the radiant is given.

Let M N (fig. 38), be a section of any deviating surface by a principal plane; D n a normal at the point D; G D an incident and D H the corresponding deviated ray. Draw any two rectangular axes as A x and A y. Call X and Y, the angles which the normal makes with the axes of x and y respectively, and we shall have

$$\sin \mathrm{X} = \cos \mathrm{Y} = \frac{d\,x}{d\,s},$$

$$\sin \mathrm{Y} = \cos \mathrm{X} = \frac{d\,y}{d\,s};$$

and calling ω and ω' the angles which the incident and deviated rays make respectively with the axis of x, we get from the triangles G D n and G′ D n,

$$\sin \varphi = \sin (\mathrm{X} - \omega),$$

$$\sin \varphi' = \sin (\mathrm{X} - \omega'),$$

or

$$\sin \varphi = \cos \omega \frac{d\,x}{d\,s} - \sin \omega . . \frac{d\,y}{d\,s}.$$

$$\sin \varphi' = \cos \omega' \frac{d\,x}{d\,s} - \sin \omega' \frac{d\,y}{d\,s};$$

these being substituted in the equation,

$$\sin \varphi - m \sin \varphi' = 0,$$

give

$$\cos \omega . d\,x - \sin \omega\, d\,y - m (\cos \omega_{,}\, d\,x - \sin \omega_{,}\, d\,y) = 0 \quad . \ (59)$$

Let $x\,y$, be the co-ordinates of the point of incidence D, $\alpha\ \beta$, those of any point on the incident, and $\alpha_{,}\ \beta_{,}$, those of any point on the deviated ray; since both of these rays pass through the point

whose co-ordinates are x and y, their respective equations will be,

$$\beta - y = \tan\omega \,.\, (\alpha - x)$$

$$\beta' - y = \tan\omega_{,}\, (\alpha_{,} - x)$$

from the first we have,

$$\frac{\overline{\beta - y}^2 + \overline{\alpha - x}^2}{\overline{\alpha - x}^2} = \tan^2\omega + 1 = \sec^2\omega = \frac{1}{\cos^2\omega}$$

or

$$\cos\omega = \frac{\alpha - x}{\sqrt{(\alpha - x)^2 + (\beta - y)^2}};$$

and

$$1 - \cos^2\omega = \sin^2\omega = \frac{\overline{\beta - y}^2}{\overline{\alpha - x}^2 + \overline{\beta - y}^2},$$

or

$$\sin\omega = \frac{\beta - y}{\sqrt{(\alpha - x)^2 + (\beta - y)^2}};$$

In the same way we obtain

$$\cos\omega_{,} = \frac{\alpha_{,} - x}{\sqrt{(\alpha_{,} - x)^2 + (\beta_{,} - y)^2}},$$

$$\sin \omega_{,} = \frac{\beta_{,} - y}{\sqrt{(\alpha_{,} - x)^2 + (\beta_{,} - y)^2}}.$$

These values in equation (59) give,

$$\frac{(\alpha - x)\,d\,x - (\beta - y)\,d\,y}{\sqrt{(\alpha - x)^2 + (\beta - y)^2}} - m\,\frac{(\alpha_{,} - x)\,d\,x - (\beta_{,} - y)\,d\,y}{\sqrt{(\alpha_{,} - x)^2 + (\beta_{,} - y)^2}} = 0 \quad (60)$$

Now if x be assumed, y becomes known from the equation of the curve of intersection of the deviating surface with the principal plane, (which we will call the *deviating curve*), and the point of incidence is, therefore, determined ; and if the radiant be also assumed, it being on the incident ray, α and β become known, and the incident ray will be given in position. These values of α β, x and y, and the differentials of the latter, deduced from the equation of the deviating curve, being substituted in equation (60), will give an equation containing only the variables $\alpha_{,}$ and $\beta_{,}$, which will be that of the deviated ray, and may be represented by

$$F\,(\alpha_{,}\,\beta_{,}) = 0 \quad . \quad . \quad . \quad (61)$$

If equation (60) be differentiated with reference to x and y, the derived equation will be a function of α β, $\alpha_{,}$ $\beta_{,}$, x y, and for the same values as before as-

sumed for the co-ordinates α β, x y, will appertain to a second deviated ray, proceeding from a point on the deviating curve whose distance from that at which the first ray is deviated, is equal to $d\,s$. This equation, which may be represented by

$$\mathrm{F}_{,}(\alpha_{,} \beta_{,}) = 0, \quad . \quad . \quad . \quad (62)$$

and equation (61), containing but two variables, viz: $\alpha_{,}$ $\beta_{,}$, these co-ordinates become known, and determine the point common to two consecutive deviated rays, which is one point in the caustic.

We have supposed the point of incidence to be given, and have found the corresponding point of the caustic; if, however, we combine equations (61) and (60) with that of the deviating curve by eliminating $x\,y$, and their differentials, the resulting equation will be independent of the co-ordinates of the points of incidence, and will be a function of α β, $\alpha_{,}$ $\beta_{,}$, and may be written,

$$\mathrm{F}_{,,}(\alpha\, \beta\, \alpha_{,}\, \beta_{,}) = 0;$$

Now, assuming the position of the radiant, or the co-ordinates α and β, the resulting equation, containing the variables $\alpha_{,}$ and $\beta_{,}$, will evidently be that of the caustic.

Surfaces of Accurate Convergence.

64. We have thus far supposed the deviation to take place at spherical surfaces, and have seen that for a pencil of any considerable magnitude, the rays at a distance from the axis wander or aberrate from the focus into which those immediately about the axis are concentrated. It is now proposed to assume a pencil of any magnitude, and to find a deviating surface which shall concentrate all its rays accurately to the same focus.

For this purpose, join the given radiant R, (fig. 39), and the point *A* into which the rays are to be collected; take this line as the axis of x, and the origin at the focus. Calling u and $u_{,}$ the distances from the point of incidence D to any two points assumed, one on the incident, the other on the deviated ray, and of which the co-ordinates are α β, and $\alpha_{,}$ $\beta_{,}$, we shall have,

$$u^2 = (\alpha - x)^2 + (\beta - y)^2$$

$$u_{,}^2 = (\alpha_{,} - x)^2 + (\beta_{,} - y)^2,$$

$$- u\, d u = (\alpha - x)\, d x + (\beta - y)\, d y.$$

$$- u_{,}\, d u_{,} = (\alpha_{,} - x)\, d x + (\beta_{,} - y)\, d y.$$

these substituted in equation (60), reduce it to

$$d u - m\, d u_{,} = 0 \quad . \quad . \quad . \quad (61)'$$

or

$$u - m\,u_{,} + n = 0 \quad . \quad . \quad . \qquad (62)'$$

Calling c, the distance between the radiant and focus, we have,

$$u = \sqrt{(c - x)^2 + y^2}$$

$$u_{,} = \sqrt{x^2 + y^2}$$

these values substituted in equation (62)′, give

$$\sqrt{(c - x)^2 + y^2} - m\sqrt{x^2 + y^2} + n = 0 \quad . \quad . \qquad (63)$$

which is the equation of a principal section of the required surface. It is of the fourth degree.

If the rays be parallel, (fig. 40), we have

$$d\,u = d\,x,$$

substituting this value of $d\,u$ in equation (61)′, we get

$$d\,x - m\,d\,u_{,} = 0$$

hence

$$x - m\,u_{,} + n = 0$$

or

$$x - m\sqrt{x^2 + y^2} + n = 0 \quad . \quad . \quad . \qquad (64)$$

Clearing the radical and reducing, we have

$$m^2 y^2 + (m^2 - 1) x^2 - 2 n x - n^2 = 0, \quad . \qquad (65)$$

which is an equation of a conic section, the nature of which will depend upon m. If m be greater than unity, it will be an ellipse ; if equal to unity, a parabola ; and less than unity, an hyperbola.

It is obvious from equation (64), that the origin is at one of the foci, since the radius vector which is equal to $\sqrt{x^2 + y^2}$, is expressed in rational functions of the absciss x. To find at which focus the origin is situated, we have, making $y = 0$,

$$x^2 - \frac{2n}{m^2 - 1} x - n^2 = 0$$

or

$$x = \frac{n}{m - 1},$$

$$x = -\frac{n}{m + 1}.$$

The positive value of x being greater than the negative, the origin is at the focus most distant from the surface of incidence.

Adding the values of x, we have the transverse

axis ; taking their difference, we have the distance between the foci ; the first is,

$$\frac{2\,m\,n}{m^2-1},$$

which we will call 2 A ; and the second

$$\frac{2\,n}{m^2-1},$$

which will be called 2 E : hence

$$\text{A} : \text{E} :: m : 1$$

or, *the semi-transverse axis is to the eccentricity, as the index of refraction to unity.*

Let M A N, (fig. 41), be the section of an ellipsoid of revolution whose semi-transverse axis and eccentricity satisfy the above proportion ; parallel rays having the direction of the axis A B will be brought to the focus F. With F as a centre, and any radius less than F A, describe the arc of a circle M N ; this will represent the section of a spherical surface to which the rays deviated at the ellipsoidal surface will be normal, and M A N M, will represent a section of an aplanatic meniscus. In this case m is greater than unity. If, however, m be

less than unity, the deviating surface becomes (fig. 42), an hyperboloid of revolution whose semi-transverse axis and eccentricity must satisfy the same proportions; the rays being parallel to the axis and incident on a plane surface perpendicular to that line, we shall have an aplanatic plano-convex lens.

If, $n=0$, the general equation (63), will reduce to

$$(m^2-1)y^2+(m^2-1)x^2+2cx-c^2=0,$$

which is an equation of a circle, (*see Davies' Anal. Geom. Book VII. Art.* 26), whose radius is

$$\frac{cm}{m^2-1}=r,$$

and the co-ordinates of whose centre are

$$x=\frac{c}{m^2-1}$$

$$y=0$$

Calling d, the distance of the radiant from the

centre of curvature, we have

$$d = c + x = c + \frac{c}{m^2 - 1} = \frac{m^2\, c}{m^2 - 1};$$

hence,

$$d : r :: m : 1$$

If, therefore, with C as a centre (fig. 43), and any radius r, we describe the arc M A N, and the distance C O, be taken bearing to r the relation given by the above proportion, rays converging to the point O, will be deviated by the spherical surface accurately to some point as F, whose distance from C is given by the value of x. With the centre F and any radius less than F A, describe the arc M S N, and the figure M A N S will be a section of an aplanatic spherical meniscus.

If in equation (63), $m = -1$, we shall have the equation of the curve of accurate convergence by reflection, which becomes after reduction,

$$4\,n^2\, y^2 - (c^2 - n^2)^2 - 4\, x^2\, (c^2 - n^2) + 6\, c\, x^2 (c^2 - n^2) = 0 \quad . \ (65).$$

This is an equation of a conic-section, the nature of which will depend upon the value of c^2 as compared with n^2. It will be that of an ellipse if $c^2 < n^2$,

or a hyperbola if $c^2 > n^2$. In case of parallel rays, equation (64) becomes, when $m = -1$,

$$y^2 - 2nx - n^2 = 0$$

an equation of a parabola.

Of the Eye and of Vision.

65. The eye is a collection of refractive media, which concentrate the rays of light proceeding from every point of an external object, on a tissue of delicate nerves, called the retina, there forming an image, from which, by some process unknown, our perception of the object arises. These media are contained in a globular envelope composed of four coatings, two of which, very unequal in extent, make up the external enclosure of the eye, the others serving as lining to the larger of these two.

The shape of the eye is spherical except immediately in front, where it projects beyond the spherical form, as indicated at $d\,e\,d''$, (fig. 44), which represents a section of the human eye through the axis in a horizontal plane. This part is called the *cornea*, and is in shape a segment of an ellipsoid of revolution about its transverse axis which coincides with the axis of the eye, and which has to the conjugate axis, the ratio 1,3. It is a strong, horny, and delicately transparent coat.

Immediately behind the cornea, and in contact with it, is the first refractive medium, called the *aqueous humour*, which is found to consist of nearly pure water, holding a little muriate of soda and gelatine in solution with a very slight quantity of albumen. Its refractive index is found to be very nearly the same as that of water, viz : 1,336, and parallel rays having the direction of the axis of the eye will, in consequence of the figure of the cornea, after deviation at the surface of this humour, converge accurately to a single point.

At the posterior surface of the chamber A, in contact with the aqueous humour, is the *iris*, *g g*, which is a circular opaque diaphragm, consisting of muscular fibres by whose contraction or expansion an aperture in the centre, called the *pupil*, is diminished or increased according to the supply of light. The object of the pupil seems to be, to moderate the illumination of the image on the retina. The iris is seen through the cornea, and gives the eye its color.

In a small transparent bag or capsule, immediately behind the iris and in contact with it, closing up the pupil, and thereby completing the chamber of the aqueous, lies the *crystalline humour* B ; it is a double convex lens of unequal curvature, that of the anterior surface being least ; its density towards the centre is found to be greater than at the edge, which corrects the spherical aberration that would otherwise exist ; its mean refractive in-

dex is 1,384, and it contains a much greater portion of albumen and gelatine than the other humours.

The posterior chamber C, of the eye, is filled with the *vitreous humor*, whose composition and specific gravity differ but little from the aqueous. Its refractive index is 1.339.

At the final focus for parallel rays deviated by these humors, and constituting the posterior surface of the chamber C, is the *retina h h h,* which is a net work of nerves of exceeding delicacy, all proceeding from one great branch O, called the *optic nerve* that enters the eye obliquely on the side of the axis towards the nose. The retina lines the whole of the chamber C, as far as *i i,* where the capsule of the crystalline commences.

Just behind the retina is the *choroid coat k k*, covered with a very black velvety pigment, upon which the nerves of the retina rest. The office of this pigment appears to be to absorb the light which enters the eye as soon as it has excited the retina, thus preventing internal reflection and consequent confusion of vision.

The next and last in order is the *sclerotic coat*, which is a thick, tough envelope $d\ d'\ d''$, uniting with the cornea at $d\ d''$ and constituting what is called the white of the eye. It is to this coating that the muscles are attached which give motion to the whole body of the eye.

From the description of the eye, and what is said in article (59), it is obvious that inverted images

of external objects are formed on the retina. This may easily be seen by removing the posterior coating of the eye of any recently killed animal and exposing the retina and choroid coating from behind. The distinctness of these images, and consequently of our perceptions of the objects from which they arise, must depend upon the distance of the retina from the crystalline lens. The habitual position of the retina, in a perfect eye, is nearly at the focus for parallel rays deviated by all the humors, because the diameter of the pupil is so small compared with the distance of objects at which we ordinarily look, that the rays constituting each of the pencils employed in the formation of the internal images may be regarded as parallel. But we see objects distinctly at the distance of a few inches, and as the focal length of a system of lenses, such as those of the eye, (equation 16′), increases with the diminution of the distance of the radiant or object, it is certain that the eye must possess the power of self adjustment, by which either the retina may be made to recede from the crystalline humor, and the eye lengthened in the direction of the axis, or the curvature of the lenses themselves altered, so as to give greater convergency to the rays. The precise mode of this adjustment does not seem to be understood. There is a limit, however, with regard to distance, within which vision becomes indistinct; this limit is usually set down at *six inches*, though it varies with different eyes. The limit here referred to is an

immediate consequence of the relation between the focal distances expressed in equation (16)′, for when the radiant or object is brought within a few inches, the corresponding conjugate or image is thrown behind the point to which the retina may be brought by the adjusting power of the eye.

With age the cornea loses a portion of its convexity, the power of the eye is, in consequence, diminished, and distinct images are no longer formed on the retina, the rays tending to a focus behind it. Persons possessing such eyes are said to be *long sighted*, because they see objects better at a distance ; and this defect is remedied by *convex* glasses, which restore the lost power, and with it, distinct vision.

The opposite defect arising from too great convexity in the cornea is also very common, particularly in young persons. The power of the eye being too great, the image is formed in the vitreous humor in front of the retina, and the remedy is in the use of *concave* glasses. Cases are said to have occurred, however, in which the prominence of the cornea was so great as to render the convenient application of this remedy impossible, and relief was found in the removal of the crystalline lens, a process common in cases of cataract, where the crystalline loses its transparency and obstructs the free passage of light to the retina.

The fact that inverted images are formed upon the retina, and we, nevertheless, see objects erect, has given rise to a good deal of discussion. With-

out attempting to go behind the retina to ascertain what passes there, it is believed that the solution of the difficulty is found in this simple statement, viz: that we look at the object, not at the image. This supposes that every point in an image on the retina, produces, without reference to its neighboring points, the sensation of the existence of the corresponding point in the object, the position of which the mind locates some where in the axis of the pencil of rays of which this point is the vertex; all the axes cross at the optical centre of the eye, which is just within the pupil, and although the lowest point of an object will, in consequence, stimulate by its light the highest point of the retina affected, and the highest point of the object the lowest of the retina, yet the sensations being referred back along the axes, the points will appear in their true positions and the object to which they belong erect. In short, instead of the mind contemplating the relative position of the points in the image, the image is the exciting cause that brings the mind to the contemplation of the points in the object.

It may be proper to remark here that the base of the optic nerve, where it enters the eye, is totally insensible to the stimulus of light, and the reason assigned for this is, that at this point the nerve is not yet divided into those very minute fibres which are capable of being affected by this delicate agent.

66. *The apparent magnitude* of an object, is determined, by the extent of retina covered by its image.

If, therefore, R R′ (fig. 45), be a section of the retina, by a plane through the optical centre C of the eye, and $AB = l$, $ab = \lambda$, the linear dimensions of an object and its image in the same plane, we shall have,

$$\lambda = \mathrm{C}a \cdot \frac{l}{\varepsilon} \quad . \quad . \quad . \quad . \quad . \quad (66)$$

calling ε, the distance of the object. And for any other object whose linear dimension is l' and distance $\varepsilon_{,}$, calling the corresponding dimension of the image $\lambda_{,}$

$$\lambda_{,} = \mathrm{C}a \cdot \frac{l'}{\varepsilon_{,}}$$

and since C a is constant, or very nearly so,

$$\lambda : \lambda_{,} :: \frac{l}{\varepsilon} : \frac{l'}{\varepsilon_{,}},$$

or the apparent linear dimensions of objects are as their real dimensions directly, and distances from the eye inversely. But $\frac{l}{\varepsilon}$, may be taken as the

measure of the angle B C A = b C a, which is called the *visual angle*, and hence the apparent linear magnitudes of objects are said to be directly proportional to their visual angles.

Small and large objects may, therefore, be made to appear of equal dimensions by a proper adjustment of their distances from the eye. For example, if $\lambda = \lambda_{,}$, we have

$$\frac{l}{\varepsilon} = \frac{l'}{\varepsilon_{,}},$$

or

$$\varepsilon_{,} = \frac{l' . \varepsilon .}{l},$$

and if $l = 1000$ feet, $\varepsilon = 20{,}000$ and $l' = ,1$ of a foot, or little more than an inch,

$$\varepsilon_{,} = \frac{20{,}000 \times ,1}{1000} = 2 \text{ feet},$$

the distance of the small object at which its apparent magnitude will be as great as that of an object ten thousand times larger, at the distance of 20,000 feet.

Microscopes and Telescopes.

67. From what has just been said, it would appear that there is no limit beyond which an object may not be magnified by diminishing its distance from the optical centre of the eye. But when an object passes within the limit of distinct vision, what is gained in its apparent increase of size, is lost in the confusion with which it is seen. If, however, while the object is too near to be distinctly visible, some refractive medium be interposed to assist the eye in bending the rays to foci upon its retina, distinct vision will be restored, and the magnifying process may be continued. Such a medium is called a *single microscope*, and usually consists of a convex lens, whose principal focal distance is less than the limit of distinct vision, and whose index of refraction is greater than unity.

To illustrate the operation of this instrument, let M N (fig. 46), be a section of a double convex lens whose optical centre is C; Q P an object in front and at a distance from C equal to the principal focal distance of the lens; E the optical centre of the eye at any distance behind the lens.

The rays Q C and P C, containing the optical centre will undergo no deviation, and all the rays proceeding from the points Q and P, will be respectively parallel to these rays after passing the lens; some rays as N E from Q, and M E from P, will pass through the optical centre of the eye, and

be the axes of two beams of light whose boundaries will be determined by the pupil, and whose foci will be at q and p on the retina, giving the visual angle,

$$\text{M}'\text{E}\text{N}' = \text{P}\text{C}\text{Q};$$

or the apparent magnitude of the object P Q, the same as if the optical centre of the eye were at that of the lens. And this will always be the case when an object occupies the principal focus of a lens whatever the distance of the eye, provided it be within the field of the rays.

Without the lens, the visual angle is Q E P < P C Q; hence the apparent magnitude of the object will be increased by the lens.

Calling λ and $\lambda_{,}$ the apparent magnitudes of the object as seen with, and without the lens, we shall have,

$$\lambda : \lambda_{,} :: \frac{\text{PQ}}{\text{CQ}} : \frac{\text{PQ}}{\text{EQ}} :: \frac{1}{\text{CQ}} : \frac{1}{\text{EQ}}.$$

or

$$\frac{\lambda}{\lambda_{,}} = \frac{1}{\text{F}_{,,}} = -(m-1)\left(\frac{1}{r} + \frac{1}{r'}\right) \quad . \quad . \quad . \quad . \quad (67)$$

by using the notation employed in equation (23), and calling E Q, the limit of distinct vision, unity.

As long as $\text{F}_{,,} < 1$, or the principal focal length

of the lens is less than the limit of distinct vision, the apparent size of the object will be increased, and the lens may be used as a single microscope.

We can now understand what is meant by the power of a lens or combination of lenses, referred to at the close of article (32). $\frac{1}{F_{\prime\prime}}$, which was there said to measure the power of a lens, we see from equation (67), expresses the number of times the apparent magnitude of an object is increased beyond that at the limit of distinct vision, by the use of the lens; and whatever has been demonstrated of the powers of lenses generally, is true of magnifying powers. Thus, in equation (22), we have the magnifying power of any combination of lenses equal to the algebraic sum of the magnifying powers taken separately. Should any of the individuals of the combination be concave, they will enter with signs contrary to those of the opposite curvature.

The power of a single microscope is, equation (67), *equal to the limit of distinct vision divided by its principal focal distance, and its numerical value will be greater as its refractive index and curvature are greater.*

68. It will be recollected that the last member of equation (67), was deduced from equation (18), by neglecting the thickness of the lens. Should, however, the microscope be an entire sphere, as is often the case, the thickness will be equal to twice

the radius and ought not to be omitted. By substituting $2\,r$ for t, equation (18) reduces to,

$$\frac{1}{f''} = \frac{m(m-1) - \frac{m\,r}{f}}{-m\,r + (m-1)2\,r - \frac{2\,r^2}{f}} - \frac{m-1}{r},$$

and supposing the rays parallel,

$$\frac{1}{F_{,,}} = \frac{m\,(m-1)}{r\,(m-2)} - \frac{m-1}{r},$$

or

$$F_{,,} = \frac{r\,(m-2)}{2(m-1)}.$$

This value of $F_{,,}$ is estimated from the first surface. When estimated from the optical centre it becomes,

$$F_{,,} = \frac{r\,(m-2)}{2\,(m-1)} - r = -\frac{m\,r}{2(m-1)};$$

and

$$\frac{1}{F_{,,}} = -\frac{2\,(m-1)}{m\,r} = -\frac{2\left(1-\frac{1}{m}\right)}{r} \quad . \quad . \quad (68).$$

from which it is obvious that the power will be greatest where m is greatest and r least.

69. To obtain a general expression for the visual angle under which the image of an object, placed at any distance from a lens, is seen, let Q P (fig.47), be an object in front of a convex lens whose optical centre is E ; $q\,p$ its image, and O the position of the eye. Calling the visual angle p O q, A, we obtain, by taking the arc equal to its tangent, the angle being very small,

$$\text{A} = \frac{q\,p}{\text{O}\,q} = \frac{q\,p}{\text{OE} - \text{E}\,q};$$

and representing the distances Q E by f; E q by f''; and E O by d, we have

$$q\,p = \text{Q P} \cdot \frac{f''}{f}$$

$$\text{E O} - \text{E}\,q = d - f''$$

hence,

$$\text{A} = \frac{\text{Q P}}{f} \cdot \frac{f''}{d - f''},$$

but $\frac{\text{Q P}}{f}$ is the visual angle, when the eye is at the centre of the lens ; calling this $\text{A}_{,}$, we have

$$\frac{\text{A}}{\text{A}_{,}} = \frac{f''}{d - f''} = \frac{1}{\frac{d}{f''} - 1}; \quad . \quad . \text{(69)}.$$

This relation has been obtained on the supposition that d and f'' are positive on the opposite side of the lens from the object; and if, as heretofore, we regard distances estimated in that direction negative, which we shall do for sake of uniformity, the equation will remain as at present written.

Now, supposing distinct vision possible for all positions of the eye, an examination of equation (69) will make it appear,

1st. That when the object is at a distance from the lens greater than that of the principal focus, in which case there will be a real image, the lens will make no difference in the apparent magnitude of the object, provided the eye is situated at a distance from it equal to twice that of the image.

2d. At all positions for the eye between this limit and the image, the apparent magnitude of the object is increased by the lens.

3d. At a position half way between this limit and the lens, the apparent magnitude of the object would be infinite.

4th. The eye being placed at a distance greater than twice that of the image, the apparent magnitude of the object will be diminished by the lens.

5th. When the distance of the object from the lens is equal to that of the principal focus, in which case f'' becomes infinite, the apparent magnitude will be the same as though the eye were situated at the centre of the lens, no matter what its actual distance behind the lens.

6th. In case of a concave lens, f'' changing its sign, the apparent magnitude of the object will always be diminished by the lens.

The visual angle when the object is placed in front of a *reflector*, (fig. 48), is found in the same way.

$$\text{A} = \frac{q\,p}{\text{O}\,q} = \frac{\text{Q P}}{\text{E Q}} \cdot \frac{\text{E}\,q}{\text{O}\,q},$$

and representing, as before, E Q, E q, and E O, by f, f', and d respectively, and the visual angle $\frac{\text{P Q}}{\text{E Q}}$ by $\text{A}_{,}$, we have

$$\frac{\text{A}}{\text{A}_{,}} = \frac{f'}{f' - d} = \frac{1}{1 - \frac{d}{f'}} \quad . \quad . \quad . \quad . \quad (70).$$

We shall not stop to discuss this equation. It may be remarked, however, that when the reflector is convex, the apparent magnitude of the object will be diminished by its use.

70. We have supposed in the preceding discussion, distinct vision to be possible for all positions of the eye ; but this we know depends upon the state of convergence or divergence of the rays. If, however, the image, when one is formed, in-

stead of being seen by the naked eye, be viewed by the aid of another lens or reflector, so placed that the rays composing each pencil proceeding from the object shall, after the second deviation, be parallel, or be within such limits of convergence or divergence that the eye can command them, the object will always be seen distinctly, and either larger or smaller than it would appear to the unassisted eye, depending upon the magnitude of the image, and the power of the lens or reflector used to view it. As most eyes see distinctly with parallel rays, this second lens or reflector is so placed that the image shall occupy its principal focus; and where this is the case, we have seen that the apparent magnitude of the image will be the same as though the eye were at its optical centre. Calling the principal focal distance of this lens or reflector $(F_{,,})$; d, in equation (69), will be $f'' + (F_{,,})$, and that equation will become,

$$\frac{A}{A_{,}} = \frac{f''}{(F_{,,})} \quad . \quad . \quad . \quad . \quad . \quad (71)$$

and if the object P Q (fig. 49), be so distant that the rays composing the small pencil whose base is M N, may be regarded as parallel, f'' becomes $F_{,,}$, and we have,

$$\frac{A}{A_{,}} = \frac{F_{,,}}{(F_{,,})} \quad . \quad . \quad . \quad . \quad . \quad (72)$$

Equation (71) involves the principles of the *compound refracting microscope*, and *refracting telescope;* and equation (72), which is a particular case of (71), relates to the *astronomical refracting telescope*. The lens M N, next the object, is called the *object or field lens*, and *m n*, *the eye lens*. The magnifying power in the first case, is equal to *the distance of the image from the field lens divided by the principal focal length of the eye lens;* and in the second, to *the principal focal length of the field lens, divided by that of the eye lens.*

If instead of a convex, a concave lens be used for the eye lens, the combination will be of the form used by Galileo, who invented this instrument in 1609. In this construction, the eye lens (fig. 50), is placed in front of the image at a distance equal to that of its principal focus, so that the rays composing each pencil shall emerge from it parallel. The rule for finding the magnifying power of this instrument is the same as in the former case; for in equation (69), we have, on account of the principal focal distance of the concave lens being of a sign contrary to that of the convex,

$$d = f'' - (-(\mathrm{F}_{,,})) = f'' + (\mathrm{F}_{,,})$$

which reduces that equation to

$$\frac{\mathrm{A}}{\mathrm{A}_{,}} = \frac{f''}{(\mathrm{F}_{,,})},$$

or for parallel rays, to

$$\frac{A}{A_{\prime}} = \frac{F_{\prime\prime}}{(F_{\prime\prime})}.$$

If we divide both numerator and denominator of equation (72), by $F_{\prime\prime} \times (F_{\prime\prime})$, it becomes,

$$\frac{A}{A_{\prime}} = \frac{\frac{1}{(F_{\prime\prime})}}{\frac{1}{F_{\prime\prime}}},$$

and calling L the power of the field, and l that of the eye lens, we have

$$\frac{A}{A_{\prime}} = \frac{l}{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad (73)$$

or the magnifying power of the astronomical telescope is equal to the quotient arising from dividing the power of the eye lens by that of the field lens.

71. If E (fig. 51), be the optical centre of the field, and O that of the eye lens of an astronomical telescope, the line E O, passing through the points E and O, is called the axis of the instrument. Let Q′ P′ be any object whose centre is in this axis, and $q' \, p'$ its image. Now, in order that all points in the object may appear equally bright,

it is obvious from the figure, that the eye lens must be large enough to embrace as many rays from the points P′ and Q′, as from the intermediate points. It is not so in the figure; a portion, if not all the rays from those points will be excluded from the eye, and the object, in consequence, appear less luminous about the exterior than towards the centre, the brightness increasing to a certain boundary, within which, all points will appear equally bright. The angle subtended at the centre of the field lens, by the greatest line that can be drawn within this boundary, is called *the field of view*. To find this angle, draw m N and M n to the opposite extremes of the lenses, intersecting the image in p and q, and the axis in X; then will $p q$ be the extent of the image of which all the parts will appear equally bright. Draw q E Q and p E P, the angle P E Q $= p$ E q, is the field of view, which will be denoted by ξ;

$$\xi = \frac{p\,q}{f''} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (74)$$

but

$$p\,q = \frac{m\,n}{\text{X O}} . \text{X}\,r \quad . \quad . \quad . \quad . \quad . \qquad (75)$$

to find X O and X r, call the diameter M N of the object lens α, that of the eye lens β, and we have

$$\alpha : \beta :: \text{E X} : \text{X O}$$

$$\alpha + \beta : \beta :: \text{E X} + \text{X O} : \text{X O}$$

hence,

$$\text{X O} = \frac{\beta}{\alpha + \beta} \cdot (f'' + (\text{F}_{,,}));$$

and in the same manner,

$$\text{E X} = \frac{\alpha}{\alpha + \beta} \cdot (f'' + (\text{F}_{,,}))$$

$$\text{X}r = f'' - \text{E X} = f'' - \frac{\alpha}{\alpha + \beta} \cdot (f'' + (\text{F}_{,,})) = \frac{\beta f'' - \alpha (\text{F}_{,,})}{\alpha + \beta};$$

these values in equation (75), give

$$p\,q = \frac{\beta f'' - \alpha\,(\text{F}_{,,})}{f'' + (\text{F}_{,,})},$$

and this in equation (74), gives, by introducing the powers of the lenses,

$$\xi = \text{L} \cdot \frac{\beta\, l - \alpha\, \text{L}}{l + \text{L}} \quad . \quad . \quad . \quad . \quad . \quad (76)$$

The rays of each of the several pencils emerging from the eye lens parallel, will be in condition to afford distinct vision, and the extreme rays $m\,\text{O}'$, and $n\,\text{O}'$, will be received by an eye whose optical

centre is situated at O′. If the eye be at a greater or less distance than O′, from the eye lens, these rays will be excluded, and the field of view will be contracted by an improper position of the eye. It is on this account that the tube containing the eye lens of a telescope usually projects a short distance behind to indicate the proper position for the eye.

From the similar triangles $p\,O\,q$ and $m\,O'\,n$, we have

$$O\,O' = \frac{m\,n}{p\,q} \cdot r\,O = \frac{\beta\,(L + l)}{\beta\,l - \alpha\,L} \cdot (F_{,,}) \quad . \ (77).$$

This also applies to the Galilean instrument, by changing the sign of l, which will render O O′, negative. The eye should, therefore, be in front of the eye glass in order that it may not, by its position, diminish the field of view; but as this is impossible, the closer it is placed to the eye glass the better.

When the telescope is directed to objects at different distances, the position of the image, (equation 19), will vary, and the distance between the lenses must also be changed. This is accomplished by means of two tubes which move freely one within the other, the larger usually supporting the object and the smaller the eye lens.

Through the common astronomical refracting telescope objects appear inverted, and through the

Galilean erect, as must be obvious on the slightest examinations of the figures of these instruments.

72. The *terrestrial telescope* is a common astronomical telescope with the addition of what is termed an *erecting piece,* which consists of a tube supporting at each end a convex lens. The length of this piece should be such as to preserve entire the field of view, and its position so adjusted that the image formed by the object glass, shall occupy the principal focus of the first lens, as indicated in figure 52. If the lenses of the erecting piece be of the same power, the magnifying power of the instrument will be equal to that of an astronomical telescope having the same object and eye lens.

73. If, now, the object approach the field lens, f'', in equation (71), will increase and the magnifying power become proportionably greater; but this would require the tube containing the eye lens to be drawn out to obtain distinct vision, and to an extent much beyond the limits of convenience if the object were very near. This difficulty is avoided by increasing the power of the object lens, as is obvious from equation (50); and when this is carried to the extent required by very near objects, the instrument becomes a *compound microscope,* which is employed to examine minute objects. The compound microscope (fig. 53), not dif-

fering in principle from the telescope, its magnifying power is given by the equation,

$$\frac{A}{A_{,}} = \frac{f''}{(F_{,,})} = \frac{1}{(F_{,,})} \cdot \frac{1}{\frac{1}{f''}};$$

and substituting for $\frac{1}{f''}$ its value in equation (23)$'''$, we have

$$\frac{A}{A_{,}} = \frac{1}{(F_{,,})} \cdot \frac{1}{\frac{1}{f} - \frac{1}{F_{,,}}};$$

or, writing D for $\frac{1}{f}$; and representing, as before, the powers of the field and eye lenses by L and l,

$$\frac{A}{A_{,}} = \frac{l}{D - L};$$

from which it is obvious that the magnifying power may be varied to any extent by properly regulating the position of the object; but a change in the position of the object would require a change in the position of the eye glass, and two adjustments would, therefore, be necessary, which would be

inconvenient. For this reason, it is usual to leave the distance between the lenses unaltered and to vary only the distance of the object to suit distinct vision. It is, however, convenient to have the power of changing the distance between the glasses, as by that a choice of magnifying powers between certain limits may be obtained, and for this purpose the object and eye glasses are set in different tubes.

74. If the field lens of the astronomical telescope be replaced by a field reflector M N, as indicated in figure 54, we have the common *astronomical reflecting telescope.* E being the optical centre, d becomes equal to $f' - (F_{,,})$, and equation (70) reduces to,

$$\frac{A}{A_{,}} = \frac{f'}{(F_{,,})},$$

and for parallel rays to,

$$\frac{A}{A_{,}} = \frac{F_{,}}{(F_{,,})}; \quad . \quad . \quad . \quad . \quad (78)$$

hence, the rule for the magnifying power is the same as for the refracting telescope.

Figure (54) represents a reflecting telescope of the simplest construction, and it is obvious that the

head of the observer would intercept the whole of the incident light, if the reflector were small, and a considerable portion even in the case of a large one ; to obviate this, it is usual to turn the axis a little obliquely, so that the image may be thrown to one side where it may be viewed without any appreciable loss of light. By this arrangement, the image would, of course, be distorted, but in very large instruments, employed to view faint and very distant objects, it is not sufficient to cause much if any inconvenience. This is Herschel's instrument.

75. The obstruction of light is in a great measure avoided in the Gregorian telescope, of which an idea may be formed from figure 55.

M N is a concave spherical reflector, having a circular aperture in the centre ; an image $p\,q$ of any distant object P Q, is formed by it as before ; the rays from the image are received by a second concave spherical reflector, much smaller than the first, by which a second image $p'\,q'$, is formed in or near the aperture of the first reflector and is there viewed through the eye lens $m\,n$. The distance of the small reflector from the first image should be greater than its principal focal distance, and so regulated that the second image be thrown in front of the eye lens, and in its principal focus. In order to regulate this distance, the small reflector is supported by a rod that passes through a longitudinal slit in the tube of the instrument, the rod being connected with a screw, as

represented in the figure, by means of which a motion in the direction of the axis may be communicated to it.

The apparent magnitudes of the images $p\,q$ and $p'\,q'$, as seen through the same eye glass at the distance of its principal focus, are as their real magnitudes; and the latter are as the distances of these images from the centre of the small reflector, article (60). But by equation (26), making $m = -1$, and recollecting that in the case before us, c is negative, we have, calling F_2, the principal focal distance of the second reflector,

$$\frac{1}{c'} = -\frac{1}{F_2} + \frac{1}{c}$$

or

$$\frac{c'}{c} = \frac{F_2}{F_2 - c};$$

hence the magnifying power of the Gregorian telescope is given by the equation,

$$\frac{A}{A_{,}} = \frac{F_{,}}{(F_{,,})} \cdot \frac{F_2}{F_2 - c} \quad . \quad . \quad . \quad (79)$$

from which it is obvious that the apparent magnitude of the object may be made as great as we please by giving a motion to the small reflector

which shall cause its principal focus to approach the first image, and drawing out, at the same time, the eye lens to keep the rays which enter the eye parallel.

76. If the small reflector be made *convex* instead of concave, we have the modification proposed by M. CASSEGRAIN, and called the *Cassegrainian telescope*, which is represented in figure 56. Its magnifying power is given by equation (79).

77. SIR ISAAC NEWTON substituted for the small curved reflector a plane one (fig. 57), inclined 45° to the axis of the instrument, and so placed as to intercept the rays before the image is formed. The state of the rays with respect to convergence or divergence not being affected by reflection at plane surfaces, the image is formed on one side, and viewed through the lens supported by a small tube inserted in the side of the main tube of the telescope. The magnifying power of the Newtonian telescope is given by equation (78).

Dynameter.

78. If any telescope, except the Galilean, properly adjusted to view distant objects, be directed towards the heavens, the field lens may be regarded as an object whose image will be formed

by the eye lens. The distance of the object in this case will be the sum of the principal focal distances or $(F_{,,} + (F_{,,}))$, and this being substituted for f, in equation (51), we get, by inverting and reducing,

$$\frac{\delta}{\delta'} = -\frac{F_{,,}}{(F_{,,})} \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

hence, *any linear dimension of the object glass of a telescope, divided by the corresponding linear dimension of its image, as formed by the eye glass, is equal to the magnifying power of the telescope.* This is the principle of the *Dynameter*, a beautiful little instrument used to measure the magnifying power of telescopes.

To understand its construction, let us suppose (fig. 58), two circular disks of *mother-of-pearl*, a tenth of an inch in diameter, to be placed one exactly over the other in the principal focus m of a lens E, and with their planes at right angles to its axis ; an image of the common centre of the disks will be formed on the retina of an eye, viewing them through the lens, at m''. If one of the disks be moved to the position m', so that its circumference be tangent to that of the other, the image of its centre will be at m''', determined by drawing from O, the optical centre of the eye, a line parallel to that joining the optical centre of the lens and the centre of the moveable disk, article (67) ; the

images will, of course, be tangent to each other, and the moveable disk will have passed over a distance equal to its diameter, viz: one tenth of an inch. We now take but one disk, and suppose the lens divided into two equal parts by a plane passing through its axis; as long as the semi-lenses occupy a position wherein they constitute a single lens, an image of the pearl will be formed as before at m''; but when one of the semi-lenses is brought in the position denoted by the dotted lines in the figure, having its optical centre at E, in a line through m, parallel to m' E, two images, tangent to each other, will again be formed; for, all the rays from the centre of the pearl, refracted by the semi-lens in this second position, will be parallel to m E$'$, and O m''' is one of these rays. It is obvious also, that the distance E E$'$, through which the moveable semi-lens has passed, is equal to the diameter of the disk of pearl.

The dynameter consists of two tubes A B, and C D, (fig. 59), moving freely one within the other, the larger having a metallic base with an aperture in the centre whose diameter is equal to one tenth of an inch, over which is placed a thin slip of *mother-of-pearl* P. In the opposite end of the smaller tube, two semi-lenses E, $E_{,,}$ are made to move by each other by means of an arrangement indicated in figure 60, wherein n is a right-handed screw with, say, fifty threads to an inch; n' is a left-handed screw, with the same number of threads, which works in the former about a com-

mon axis, and is fastened to the frame that carries the semi-lens E. The screw n, is rendered stationary as regards longitudinal motion, by a shoulder that turns freely within the top of the frame S T at r, and works in a *nut* at v connected with a frame that carries the semi-lens E′; this screw is provided with a large circular head X Y, graduated into one hundred equal parts, which may be read by means of an index at X or Y, on the frame of the instrument. At t is a spring that serves to press the frames against their respective screws, to prevent loss of motion when a change of direction in turning takes place.

When the graduated head is turned once round to the right, the semi-lens E′, is drawn up $\frac{1}{50}$ of an inch, while the semi-lens E, is thrust in an opposite direction through the same distance, making in all a separation of the optical centres of $\frac{1}{25}$ of an inch, and the lens is kept symmetrical with regard to the centre of the instrument. If the screw had been turned through but one division on the head, the separation would have been $\frac{1}{100}$ of $\frac{1}{25}$ or $\frac{1}{2,500}$ of an inch.

To u e the instrument, direct the telescope, whose power is to be measured, to some distant object, as a star, and adjust it to distinct vision; turn it off the obje t, and apply the dynameter with the pearl end next the eye lens, and an image of the object lens will be seen; turn the graduated head, supposed to stand at zero, till two images appear and become tangent to each other; read

the number of divisions passed over, and multiply it by $\frac{1}{2,500}$, the product will give the diameter of the image in inches. Measure by an accurate scale, the diameter of the visible portion of the object glass, which being divided by the measure of its image just found, will give the magnifying power. The index will indicate zero, if the dynameter be properly adjusted, when the semi-lenses have their optical centres coincident.

This little instrument is the more valuable, because it gives, by an easy process, the magnifying power of any telescope, having a convex eye lens, however complicated. It will not apply to the Galilean telescope, because the eye lens is concave and no image of the object lens is, in consequence, formed by it.

Micrometer.

79. When a telescope is used for certain astronomical purposes, it is usual to put a number of fine wires or spiders' webs, at the focus of the object glass, to determine when any object, as a star, is in the axis of the instrument. These constitute what is called a *micrometer,* which in its simplest form is represented in figure 61. A B, is a circular diaphragm divided into equal parts by five parallel wires, all of which are bisected at right angles by a sixth. The diaphragm is so placed in the telescope that the point O, being the intersec-

tion of the sixth with the middle one of the five parallel wires, shall coincide with its axis. If now, the telescope be directed to a body moving through the field of view in the direction indicated by the sixth wire, the time of its passing each of the parallel wires may be noted, and a mean of the five observations will give the approximate time of the body's passing the axis of the instrument.

Another kind of wire micrometer is often used with the telescope to measure very small angles. It consists of two wires a, c, (fig. 62), which are made to move parallel to each other by means of fine screws A, C, each screw carrying a fork A′, C′, to which the wires are attached. The screws have fifty threads to the inch, and are provided with large circular heads graduated into 100 equal parts each, so that a turn through one division on the head, will cause the wire connected with it to pass through a distance of $\frac{1}{5,000}$ of an inch. A third wire, perpendicular to the two first, is supported by a small diaphragm, disconnected with the screws, upon one of the interior edges of which is placed a graduated scale in the shape of saw teeth to indicate the number of entire revolutions of each screw, the instrument being so adjusted that the index of each head shall mark zero, when the wires coincide with each other, and accurately bisect a small circular hole in the stationary diaphragm immediately under the middle tooth of the scale.

To ascertain the *angular* value of one division

on the screw head, find by trigonometrical computation the angle subtended by any distant and well defined object; direct the telescope, with the micrometer in its place, upon it, and adjust to distinct vision; turn the graduated heads till the object is accurately embraced by the wires, and count the number of divisions passed over by each head; add these together and divide the angle reduced to seconds by the sum, the quotient will give the value sought. If the object be so near, however, that the rays received from it may not be regarded as parallel, a correction will be necessary. To view near objects, the eye lens must be drawn out, in which case the telescope, equation (71), will have an increased magnifying power with a corresponding decrease in the value of the micrometer revolution. But the magnifying power when the image is in the principal focus, is to that when in any other [position, as $F_{\prime\prime}$ to f''; equations (71) and (72).

Calling e, the distance of the image from the principal focus, we have

$$e = f'' - F_{\prime\prime} = \frac{f F_{\prime\prime}}{f - F_{\prime\prime}} - F_{\prime\prime} = \frac{F^2_{\prime\prime}}{f - F_{\prime\prime}}.$$

and

$$F_{\prime\prime} : F_{\prime\prime} + e :: a : x,$$

a, representing the approximate value found by the first process, and x the true value.

Example. The length of the object was three feet, measured in a direction perpendicular to the line of sight; the distance from the object glass 261,9 yards; the principal focal length of the object glass, 45,75 inches, and the sum of the divisions passed over by the screw heads 1819. Call the angle subtended y.

1st. $\text{Tan}\,\frac{1}{2}\,y = \frac{\text{R.}\ ,5^{\text{yds.}}}{261,9^{\text{yds.}}}$, the log. of which is 7·280835,

and

$$y = 13'.07''.57 = 787'',57$$

hence,

$$a = \frac{787,57}{1819} = 0'',433.$$

2d. $$e = \frac{\text{F}^2{}_{\prime\prime}}{f - \text{F}_{\prime\prime}} = \frac{1,6493}{261,9 - 1,2708} = 0.,0062;$$

then,

$1,2708 : 1,2770 :: 0'',433 : 0'',435$, the true value of one division on the screw head.

The micrometer is usually provided with several eye lenses, the object of which is to increase or diminish the field of view as well as to regulate the magnifying power of the telescope. A change

in the eye lens will not affect the value of the micrometer revolution, because the apparent motion of the wires will undergo the same change as the apparent magnitude of the image. But if the object glass be changed, or the micrometer be applied to a different telescope with the same eye lens, the value of the revolution will be altered, and it will be equal to its value in the first telescope, multiplied into the ratio of the magnifying powers of the telescopes, taken inversely. The magnifying powers may be easily found by the dynameter.

The Sextant.

80. This instrument is also employed to measure angles, but on a much larger scale. It depends upon the catoptrical principle explained in article (26), and consists essentially of two reflectors I and H, (fig. 63), which stand at right angles to the plane of the instrument, in which is a graduated arc A B, of sixty degrees, represented in the plane of the paper ; a moveable index and vernier ; and the frame work necessary to support these in their position, and keep the instrument steady. A telescope T, having its optical axis, or line of collimation, as it is called, parallel to the plane of the graduated arc, and six colored glasses, of different shades, three at G, and three at G′, are added. The colored glasses are susceptible of a motion in

their own planes, and at right angles to that of the instrument, about hinges at n and n'. The purpose of the telescope is to magnify and define the objects whose angular distance is to be taken, and the colored glasses to qualify their light.

The reflector I, called the *index glass*, is attached to the index arm IV, which is moveable about the centre of curvature of the graduated arc as a centre, and is made of glass ground perfectly plane with its posterior surface, (that next the eye at E), covered with an amalgam of tin and mercury; the reflector H, called the *horizon glass*, is also plane, having half its anterior surface covered, the line separating the covered from the transparent half being parallel to the plane of the instrument, and the latter half lying to the left as indicated by the position of the eye. The telescope is supported by a ring S, attached to a stem, called the *up and down* piece, which admits of a motion, by means of a milled screw, perpendicular to the line of collimation, the purpose of which motion is to render an object seen through the transparent part, and another seen by reflection from the covered part of the horizon glass, equally bright, by bringing the telescope in a position such that nearly the same number of rays may be received from each.

Now, a ray of light XI, from the top of a steeple, for example, being incident upon the index glass, is reflected to the horizon glass in the direction IH, and by the latter to the eye in the direction

H E, through the telescope, at the same time that a ray reaches the eye in the same direction through the transparent part of the horizon glass, from the point Y ; so that the points X and Y will seem to occupy the same position in space. X E Y, is the angle subtended at the eye by the distance X Y ; but this angle being that made by the direct ray X I and the same ray after two reflections, is, article (26), double the angle H M I, made by the reflectors. I D being drawn parallel to H M, D I M = H M I will be half the angle subtended by the object. If, therefore, the angle D I F = 60°, be divided into 120 equal parts, and these be numbered as whole degrees, beginning at the line I D, and the zero of the vernier V be placed in the plane of the index glass produced, the reading of the instrument will indicate the entire angle X E Y. To observe with a sextant, then, it is only necessary to hold the plane of the instrument in that of the objects and the eye, and cause, by a motion of the index arm, the objects apparently to coincide.

Of the Adjustments.

81. The objects of the principal adjustments are : 1st, to make the index and object glasses perpendicular to the plane of the instrument ; 2d, to make these glasses parallel when the zero of the vernier coincides with that of the graduated arc ; and

3d, to make the line of collimation parallel to the plane of the instrument.

To accomplish the first, move the index division of the vernier to the middle of the graduated arc, or limb, as it is called ; then holding the instrument horizontal with the index glass towards the observer, look obliquely down the index glass so as to see the circular arc by direct view and by reflection at the same time. If the arc appear broken, the position of the glass must be altered till it appear continuous, by means of small screws that attach the frame of the glass to the instrument. The horizon glass is known to be perpendicular to the plane of the instrument when, by a sweep of the index, the reflected image of an object and the image seen directly, pass accurately over each other; and any error is rectified by means of an adjusting screw, provided for the purpose, at the lower part of the frame of the glass.

The second adjustment is effected by placing the index or zero point of the vernier to the zero of the limb; then directing the instrument to some distant object, (the smaller the better), if it appear double, the horizon glass must, after easing the screws that attach it to the instrument, if there be no adjusting screw for the purpose, be turned around a line in its own plane and perpendicular to that of the instrument, till the object appear single ; the screws being tightened, the perpendicular position of the glass must again be examined. This adjustment may be rendered unne-

cessary by correcting an observation by what is called the *index error*, which is equal to the angular separation of the two images of a single object when the zero of the vernier and that of the limb coincide; to find its value, move the index till the images run into each other and appear as one; the arc from zero of the limb to that of the vernier will be the index error. This may sometimes be measured on the arc O A, over which the graduation is continued for that purpose, and is said to be measured *off the arc*, or it may be measured on the arc O B, when it is said to be measured *on the arc*. In the first case, it is obvious the index error should always be added to the observed angle, and in the second subtracted. A better way, perhaps, to find the index error, is, to turn the instrument on any object, as the sun, for example, and cause the images of that body to be tangent to each other with the index *on* the arc, then with the index *off* the arc; the half difference of the readings will be the index error which will be positive or negative, according as the latter or former reading is the greater.

Example.

Reading on the arc	$-31'.56''$
off	$+31.22$
	$2).34''$
Index error	$-0'.17''$

The third adjustment is made by the aid of two parallel wires placed in the common focus of the telescope for the purpose of directing the observer to the centre of the field of view, in which, an observation should always be made; these wires are parallel to the plane of the instrument, and divide the field of view into three nearly equal parts. The sun and moon are made tangent to each other, when their angular distance is 90° or more, at one of the wires; the position of the sextant is then altered so as to bring these bodies to the second wire; if the contact continue, the line of collimation is parallel to the plane of the instrument; if not, the position of the telescope must be altered by means of two adjusting screws connected with the up and down piece.

It is important that the index glass, and the colored glasses used with it, be perfectly plane; for if the faces of each of these be not parallel, the observations will be fallacious, the amount of error depending upon the angle of the faces and the inclination of the incident ray. A sextant in which these glasses are defective is not, however, entirely useless, because an accurate result may be obtained by repeating the observation with the glasses revolved in their own planes through an angle of 180°, and taking a mean of the two readings. The horizon glass, with its shaded glasses, should also be plane, though this is perhaps less important, because that glass being stationary and the telescope also, the line of sight and the sur-

faces of this glass preserve the same inclination to each other.

The Artificial Horizon.

82. To measure directly the altitude of any celestial object with the sextant, it would be necessary that the object and horizon should both be distinctly visible, but this is not always the case, in consequence of the irregularity of the ground which frequently conceals the horizon from view. The observer is, therefore, obliged to have recourse to an *artificial horizon*, which consists usually of the reflecting surface of some liquid, as that of mercury, contained in a small vessel A, (fig. 64), which will arrange itself parallel to the natural horizon D A C. A ray of light S A, from a star at S, being incident on the mercury at A, will be reflected in the direction A E, making the angle $S A C = C A S'$, ($A S'$ being E A produced), and the star will appear to an eye situated at E, as far below the horizon as it is actually above it. Now with a sextant whose index and horizon glasses are represented at I and H, the angle $S E S'$, may be measured; but $S E S' = S A S' - A S E$, and because A E is exceedingly small compared with the distance A S, of any celestial object, the angle A S E may be neglected, and $S E S'$ will equal $S A S'$, or double the altitude of the star; hence one half the reading

of the instrument will give the apparent altitude. At sea, the observer has the natural, or sea horizon as a point of departure, and the altitude may be measured directly.

Camera Lucida.

83. This little instrument, the invention of Dr. Wollaston, is of great assistance in drawing from nature. In its simplest form, it consists of a glass prism, a section of which is represented by ABCD, (fig. 65), with one right angle at A, and the opposite angle C, 135°. Rays proceeding from a point of any object S in front of the face A D, enter this face without undergoing any material deviation, and being received in succession by the faces D C and C B within the limits of total reflection, they are reflected, and finally leave the face B A, in nearly the same state of divergence as when they left the object S. The eye E, being so placed that the edge B of the prism shall bisect the pupil, will receive the rays from the prism and bring them to a focus *r*, on the retina, at the same time that it will receive through the half of the pupil not covered by the prism, rays proceeding from the point P of a pencil placed below on a sheet of paper, and bring them to the same focus *r*; so that the point in the object and point of the pencil will appear to coincide on the paper, the whole of which will be seen through the uncovered

half of the pupil, and a picture of the object may thus be traced by bringing the pencil in succession in apparent contact with its various parts.

The linear dimensions of the picture will be to those of the object, as the distance of the camera from the paper, to its distance from the object, nearly.

If the paper be very near, the eye may not have the power to bring the rays proceeding from the pencil to the same focus with those from the object; this difficulty is obviated by the use of a convex lens at L, or a concave one at L′; the effect of the former being to reduce the divergence of the rays from the pencil to the same degree with that of those from the object, and of the latter, to increase the divergence of the rays from the object, and render it the same with that of the rays from the pencil. The camera lucida is constructed of various forms, having reference to the facility of using it, the optical principle being the same in all.

Camera Obscura.

84. This instrument is also used to copy from nature, and like the camera lucida, is constructed in various ways, one of the best of which, is represented by fig. 66. A B C is a *prismatic lens,* which is nothing more than a triangular prism with one or both of its refracting faces ground to spherical surfaces; it is set in a small box resting on a

cylindrical tube *t v*, that moves freely in a similar tube in the top of a dark chamber, formed by uprights or legs, about which, is suspended a cotton cloth rendered impervious to light by some opaque size. On one face of the box *m n*, containing the prismatic lens, is an opening to admit the light from any object in front of the instrument, and on one side the cloth has been omitted in the figure to show a table X Y, supported by the uprights, on which the paper is placed to receive the picture. Now, the rays from any point in an object S, will enter the face A C of the prismatic lens, be totally reflected by A B, and brought by C B, to a focus on the paper, from which, owing to the minute irregularities of its surface, they will be reflected in all directions; and thus a picture of the external object S will be painted at S′, which may easily be traced by a person situated within the folds of the cloth forming the dark chamber. The effect of the prismatic lens being the same as a convex lens, except that the former also changes the direction of the axis of a pencil deviated by it, it is obvious that the surface of the paper should be spherical. The image of the object is brought accurately to the table X Y, by means of the tube *t v*, which admits of a vertical motion in the top of the chamber; this tube also admits of a horizontal motion, the purpose of which is, to take in different objects in succession without changing the position of the body of the instrument.

The Magic Lantern.

85. This consists of a small close chamber (fig. 67), from one side of which proceeds a tube containing usually two convex lenses A and B, with an intermediate opening for a glass slide C which may be moved freely in a direction at right angles to the common axis of the lenses. Within the chamber is an Argand lamp D, behind which is a concave reflector E. The rays proceeding from any point in a figure, painted with some transparent pigment upon the glass slide and strongly illuminated by the lens A upon which the direct light from the lamp, as well as that from the reflector E is concentrated, will be brought to a focus by the lens B, on a screen M N, placed at a distance in front of the instrument; here the rays being reflected will proceed as from a new radiant, and a magnified image of the figure will thus appear upon the screen. Should the screen be partially transparent, a portion of the rays will be transmitted, and the image will be visible to an observer behind it.

The linear dimensions of the object or figure, will be to those of the image, as their respective distances from the lens B; if, therefore, the lens B be mounted in a tube which admits of a free motion in that containing the lens A, its distance from the figure may be varied at pleasure,

and the image on the screen made larger or smaller; the instrument, at the same time, being so moved as to keep the screen in the conjugate corresponding to the focus occupied by the glass slide. The instrument with an arrangement by which this can be accomplished is called, the *phantasmagoria*. In order, however, that the deception may be complete, there must be some device to regulate the light, so that the illumination of the image may be increased with its increase of size, not diminished, as it would be without such contrivance.

Solar Microscope.

86. This is the same as the magic lantern, except that the light of the sun is used instead of that from a lamp. D E (fig. 68), is a long reflector on the outside of a window shutter, in which there is a hole occupied by the tube containing the lenses.

The object to be exhibited is placed near the focus of the illuminating lens A, so as to be perfectly enlightened and not burnt, which would be the case were it at the focus.

Unequal Refrangibility of Light.

87. We have hitherto regarded light as it comes from the sun or any self-luminous body, as a simple principle, and supposed that all its integrant

parts are subject to the same laws of deviation, and have the same index. But this is not the case; for, if a beam S S′ (fig. 69), of solar light be admitted into a dark room through a small hole, in a window shutter, for example, and received upon a screen XY, it will exhibit a round, luminous spot at T, in the direction of S S′ produced; but if the face of a refracting prism A B C be interposed, the spot T will disappear, and there will be formed upon the screen above, an elongated image of the sun variously and beautifully colored, beginning with *red* on the end in the direction of the refracting angle A of the prism, and passing in succession through *orange*, *yellow*, *green*, *blue*, *indigo*, and terminating in *violet*, making seven in all. These colors are not separated by well defined boundaries, but run imperceptibly into each other; nor are the colored spaces of the same length. The following table exhibits the relative lengths of these spaces as obtained by Sir Isaac Newton with the glass prism used by him, and by Fraunhofer, with a prism made of flint glass.

	Newton.	Fraunhofer.
Red	45	56
Orange	27	27
Yellow	47	27
Green	60	46
Blue	60	48
Indigo	48	47
Violet	80	109
Total length	360	360

This colored image, called the *solar spectrum*, is accounted for, by supposing white light to be composed of an almost infinite variety of elements differing from each other in the degree with which they are deviated by refraction, and that these elements are divided into seven classes distinguished by the colors of the spectrum, the elements of each class also differing from each other, within certain limits, in the amount of their deviation.

From equation (8), we have,

$$m = \frac{\sin (a + \delta)}{\sin a} \quad . \quad . \quad . \quad (81)$$

in whice a, is the refracting angle of the prism, and δ the angle of deviation; and from which. because a is constant, we may obtain,

$$d\,m = \frac{\cos (a + \delta)}{\sin a} \,.\, d\,\delta.$$

or

$$d\,\delta = \frac{\sin a}{\cos (a + \delta)}\, d\,m \quad . \quad . \quad . \quad . \quad (82).$$

From this equation it is obvious, that as m varies, δ also varies, and that we have only to attribute to m, different values within certain limits, beginning with the red rays for which it is

least, and terminating with the violet for which it is greatest, to obtain the deviation necessary to embrace the entire spectrum.

Let A be a refracting prism (fig. 70), made of any transparent medium, with its edges placed horizontally ; $m\,n$ a graduated circle, to the centre of which a small telescope is attached in such a manner that its line of collimation shall move in a plane parallel to that of the graduated circle, which is held in a position at right angles to the edges of the prism. The telescope, being provided at a solar focus with a fine wire perpendicular to the plane of the circle, is directed to some distant luminous object, and the reading of the vernier noted. It is then directed so as to receive the colored rays from the prism, and the reading again noted when the prism is turned to the position giving the deviation a minimum. We shall then have

$$\mathrm{R\,D\,C} = \delta = \mathrm{D\,C\,S} + \mathrm{D\,S\,C}$$

or neglecting the very small angle subtended by D C at the distance of the object,

$$\delta = \mathrm{D\,C\,S},$$

which is the difference of the readings ; and this in equation (81), will give the value of m.

If the color occupying the middle of the spectrum be taken, we shall find the value of m which answers to what is called the mean deviation, and which is the same as that given in the table of article (19).

This property of white light, by which its several elements give different indices of refraction with the same medium, is called its *unequal refrangibility*.

If a hole be made in the screen (fig. 69), at any one of the colors, as green, for example, and this color, after passing through, be deviated by a second prism P', no further decomposition will be found to take place, but a green image, of the shape and size of the hole in the first screen, will be formed upon a second screen held behind at G'; and this being true of every part of the spectrum, each of the seven colors, is said to be *homogeneous* or *primary*.

The colors of the spectrum being received, each upon a separate mirror, (fig. 71), may, by varying the relative position of the mirrors, be reunited, by reflection, on a screen at W when a white spot will be formed as though it were illuminated with common light.

Dispersion of Light.

88. From what has already been said, it is obvious, that white light may be *decomposed*, *analyzed*,

or separated into its elementary colors, by refraction. The act of such separation is called, the *dispersion of light*, and that property of any medium by which this is performed is called, its *dispersive power*.

Supposing the incident beam perpendicular to the first face of the prism, the angle of incidence on the second will be equal to its refracting angle (23); and calling ψ the angle of emergence, there will be, because of the relation

$$\sin \psi = m \sin a, \quad . \quad . \quad . \quad . \quad . \quad (83)$$

$$\delta = \psi - \alpha,$$

an increase in δ, for every increase of a from zero to the limit of total reflection. It is evident, therefore, that, by supposing m to vary, two prisms may be made of different media, whose refracting angles shall be so related as to give the same mean refraction; but when this is the case, it is rarely found that the lengths of the spectra will be equal, the *red* rays being more, and the *violet* less refracted in one than in the other. The dispersive power of the medium of which the prism giving the longer spectrum is made, will, therefore, be the greater. If the refracting angle of this prism be diminished, the length of its spectrum will be diminished as well as its mean deviation, so that, the lengths of the spectra may become equal, when the

mean deviations are very unequal. It is, hence usual to take as a measure of the dispersive power of any medium, *the angle* R n V, (fig. 72), *subtended by the spectrum, divided by the angle* T n G, of *mean deviation.* From equation (83) we have

$$m - 1 = \frac{\sin \psi - \sin a}{\sin a},$$

and if the angles ψ and a be small,

$$m - 1 = \frac{\psi - a}{a}:$$

denoting by m_v, m_r and m, the indices corresponding to the extreme violet, extreme red and middle rays respectively, we shall have,

$$m_v - 1 = \text{T}\,n\,\text{V}\,\frac{1}{a},$$

$$m_r - 1 = \text{T}\,n\,\text{R}\,\frac{1}{a},$$

$$m - 1 = \text{T}\,n\,\text{G}\,\frac{1}{a},$$

and calling D, the dispersive power, we obtain, according to the rule just given,

$$\text{D} = \frac{(m_v - 1) - (m_r - 1)}{m - 1} = \frac{m_v - m_r}{m - 1} \quad . \quad . \quad (84).$$

In this way the dispersive powers of the substances named in the following table, as well as those of a great many others, have been obtained, having previously found the indices of refraction for the extreme and middle rays of the spectrum formed by each substance.

Table of Dispersive Powers.

Substances.	$\frac{m_v - m_r}{m - 1}$	$m_v - m_r$
Realgar melted,	0.267	0.394
Chromate of Lead,	0.262	0.388
Oil of Cassia,	0.139	0.089
Flint Glass,	0.050	0.032
Crown Glass,	0.033	0.018
Olive Oil,	0.038	0.018
Water,	0.035	0.012
Muriatic Acid,	0.043	0.016

89. There is a circumstance connected with this subject which should be carefully noticed, owing to its importance in the construction of lenses. If the lengths of spectra formed by two prisms of different media be the same, the colored spaces in the one will not, in general, be equal in length to the corresponding spaces of the other. This circumstance has been called the *irrationality of dispersion.*

Chromatic Aberration.

90. It follows from the unequal refrangibility of the elements of white light, that the action of a lens, (fig. 73), will be, to separate these elements and direct them to different foci, since the value of f'', in equation (19), depends upon that of m. Substituting in that equation $-\frac{1}{\varrho}$ for $-\left(\frac{1}{r}+\frac{1}{r'}\right)$, in the case of a double convex lens; and writing f_v, and f_r for the focal distances of the violet and red rays, we obtain

$$\frac{1}{f_v} = -(m_v - 1).\frac{1}{\varrho} + \frac{1}{f}$$

$$\frac{1}{f_r} = -(m_r - 1).\frac{1}{\varrho} + \frac{1}{f}$$

in which m_v, being greater than m_r, f_v, will be less than f_r, and the violet rays will be brought to a focus soonest. This deviation from accurate convergence, caused by the unequal refrangibility of the elements of white light, when deviated by a lens, is called *chromatic aberration*, and depends upon the nature of the lens and not on its figure. It is measured, along the axis of the lens, by the value of $f_r - f_v$.

The intersection of the cone of violet rays, with that of the red rays, will give what is called, the *circle of least chromatic aberration.* The diameter and position of this circle can readily be found. From the point s, (fig. 73), demit the perpendicular $s\,\mathrm{O} = y$ to the axis; this will divide $f_r - f_v$, into two parts $\mathrm{V\,O} = x$, and $\mathrm{O}\,r = w$; and calling the semi-aperture of the lens a, we shall obtain from the similar triangles of the figure,

$$\frac{y}{a} = \frac{w}{f_r} = \frac{x}{f_v}.$$

whence we deduce

$$w + x = f_r - f_v = \frac{y}{a}(f_r + f_v)$$

or

$$y = a\,\frac{f_r - f_v}{f_r + f_v} \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

The denominator of this expression is equal to twice the mean value of f'', and therefore,

$$2\,y = a\,(f_r - f_v) \cdot \frac{1}{f''}$$

and from equation (19), we have

$$\frac{1}{f_v} - \frac{1}{f_r} = \frac{m_v - 1}{\varrho} - \frac{m_r - 1}{\varrho} = \frac{m_v - m_r}{\varrho},$$

or

$$f_r - f_v = \frac{m_v - m_r}{\varrho} \cdot f''^2$$

by substituting f''^2, for $f_r \cdot f_v$, to which it is nearly equal.

Substituting the value of ϱ, from second equation of group (A), the above becomes

$$f_r - f_v = \frac{m_v - m_r}{m - 1} \cdot \frac{f''^2}{F_{\prime\prime}}.$$

hence,

$$2y = a \cdot \frac{m_v - m_r}{m - 1} \cdot \frac{f''}{F_{\prime\prime}} = a \cdot D \cdot \frac{f''}{F_{\prime\prime}} \quad . \quad . \quad (86)$$

In the case of parallel rays, the last factor is unity, from which we conclude, that *the diameter of the least circle of chromatic aberration is equal to the semi-aperture of the lens, multiplied by the dispersive power.*

The distance of this circle from the lens is,

$$f_v + x = f_v + \frac{f_v \cdot y}{a},$$

replacing $\frac{y}{a}$ by its value in equation (85), we have

$$f_v + x = \frac{2 f_v f_r}{f_r + f_v} \quad . \quad . \quad . \quad . \quad (87).$$

The effect of chromatic aberration is to give color to the image of an object, and to produce confusion of vision in consequence of the different degrees of convergence in the differently colored rays proceeding from the same points of an object. The vertices of the cones composed of these rays, lying in the axis, every section perpendicular to this line will have its brightest point in the centre, and the yellow rays converging nearly to the mean focus, and having by far the greatest illuminating property, the bad effects which would otherwise arise from this aberration are in part destroyed. Besides, these effects may be lessened by reducing the aperture of the lens, though not in the same degree as those arising from spherical aberration.

Achromatism.

91. It is, then, impossible, by the use of a sin-

gle homogeneous lens, to deviate a beam or pencil of white light accurately to a single focus, and, consequently, impossible, by the use of such a lens, to form a colorless image of any object; both, however, may be done by the union of two or more lenses of different dispersive powers. The principle according to which this may be accomplished, is termed *Achromatism*, and the combination is said to be *achromatic*.

Let us suppose two lenses of different dispersive powers placed close together, the power of the combination will, equation (22), for any one of the elementary colors as red, be

$$\frac{1}{F_r} = \frac{m_r - 1}{\varrho} + \frac{m_{r'} - 1}{\varrho'}$$

and for violet,

$$\frac{1}{F_v} = \frac{m_v - 1}{\varrho} + \frac{m_{v'} - 1}{\varrho'}$$

If F_r and F_v, were equal, the chromatic aberration, as regards these colors, would be destroyed; equating them we have,

$$(m_r - 1)\,\varrho' + (m_{r'} - 1)\,\varrho = (m_v - 1)\,\varrho' + (m_{v'} - 1)\,\varrho$$

whence,

$$\frac{\varrho}{\varrho'} = \frac{(m_v - 1) - (m_r - 1)}{(m_{r'} - 1) - (m_{v'} - 1)} = -\frac{m_v - m_r}{m_{v'} - m_{r'}},$$

the second member being negative because m_v, is greater than m_r.

Multiplying both members of this equation by $\frac{m_, - 1}{m - 1}$, it may be put under the form,

$$\frac{\frac{m_, - 1}{\varrho'}}{\frac{m - 1}{\varrho}} = -\frac{\frac{m_v - m_r}{m - 1}}{\frac{m_{v'} - m_{r'}}{m_, - 1}} \quad . \quad . \quad (88)$$

The second member expresses the ratio of the dispersive powers of the media, and the first the inverse ratio of the powers of the lenses for the mean rays; this being negative, one of the lenses must be concave the other convex, and the powers of the lenses being inversely as their focal distances, we conclude, that *chromatic aberration, as regards red and violet, may be destroyed by uniting a concave with a convex lens, the principal focal lengths being taken in the ratio of their dispersive powers.*

The usual practice is to unite a convex lens of crown glass with a concave lens of flint-glass, the focal distance of the first being to that of the

second as 33 to 50, these numbers expressing the relative dispersive powers as determined by experiment. The convex lens should have the greatest power, and, therefore, be constructed of the crown-glass ; otherwise, the effect of the combination would be the same as that of a concave lens with which it is impossible to form a real image.

To illustrate, let parallel rays be received by the lens A (fig. 74) ; its action alone would be, to spread the different colors over the space V R, whose central point *m* is distant from A, 33 units of measure, (say inches), the violet being at V and red at R ; the action of the lens B alone would be, to disperse the rays as though they proceeded from different points of the line $V' R'$, equal to V R, whose central point m', is distant from $B = 50$ inches, the violet appearing to proceed from V' and red from R' ; and the effect of the united action will be, to concentrate the red and violet at $\frac{R}{V}$, whose distance from the lens is equal to the value of F, deduced from the formula

$$\frac{1}{F} = \frac{1}{33} - \frac{1}{50} = \frac{1}{97.06} \text{ inches,}$$

or

$$F = 97.06 \text{ inches.}$$

Now if any one of the colors, orange for ex-

ample, at O, in the space R V, were thrown by the convex lens just as far in advance of the centre m, as the same color at O′ in the space V′ R′, is thrown by the concave lens behind the centre m', it is obvious that this color would also be united with the violet and red at $\frac{R}{V}$, by the joint action of both lenses; and the same would be true of any other color. But owing to the *irrationality of dispersion* of the media of which these lenses are composed, no such union can take place, the mean value of m' O′, being greater than that of m O; hence this color will not be united with the red and violet, and the distance from the point $\frac{R}{V}$, at which it will be thrown, will be equal to $(m'_{o'} - m_o)$, laid off towards the lenses. The same being true of the remaining colors, except as regards the distance at which they are found, some being to the right, others to the left of $\frac{R}{V}$, it follows, that an image formed by such a combination of lenses will be fringed with color; and this is found to be the case in practice, the colors of the fringe constituting what is called *a secondary spectrum.* An additional lens is sometimes introduced to complete the achromaticity of this arrangement.

92. If two lenses, constructed of media between

which there is no irrationality of dispersion, be united according to the conditions of equation (88), the combination would be perfectly achromatic. It is found that between a certain mixture of muriate of antimony with muriatic acid, and crown-glass, and between crown-glass and mercury in a solution of sal ammoniac, there is little or no irrationality of dispersion. These substances have therefore been used in the construction of compound lenses which are perfectly achromatic. Figure (74)′ represents a section of one of these, consisting of two double convex lenses of crown-glass, holding between them, by means of a glass cylinder, a solution of the muriate in the shape of a double concave lens, the whole combined agreeably to the relations expressed by equation (88). The focal distance of the convex lenses is determined from equation (22).

93. From equation (86) we infer, that the circle of least chromatic aberration is independent of the focal length of the lens, and will be constant, provided, the aperture be not changed. Now, by increasing the focal length of the object glass of any telescope, the eye lens remaining the same, the image is magnified ; it follows, therefore, that by increasing the focal length of the field lens, we may obtain an image so much enlarged that the color will almost disappear in comparison. Besides an increase of focal length, is attended with a diminu-

tion of the spherical aberration. This explains why, when single lenses only were used as field glasses, they were of such enormous focal length, some of them being as much as a hundred to a hundred and fifty feet. The use of achromatic combinations has rendered such lengths unnecessary, and reduced to convenient limits, instruments of much greater power than any formerly made with single lenses.

Absorption of Light.

94. If a beam of white light be received upon any medium of moderate thickness, it will, in general, be divided into three parts, one of which will be reflected, another transmitted, and the third lost within the medium, or as it is termed, *absorbed*. The quantity absorbed is found to vary not only from one medium to another, but also in the same medium for the different colors; this will appear by viewing the prismatic spectrum through a plate of almost any transparent colored medium, such as a piece of *smalt blue glass*, when the relative intensity of the colors will appear altered, some colors being almost wholly transmitted, while others will disappear or become very faint. Each color may, therefore, be said to have, with respect to every medium, its peculiar *index of transparency* as well as of refraction.

The quantity of each color transmitted, is found to depend, in a remarkable degree, upon the thickness of the medium, for, if the glass just referred to be extremely thin, all the colors are seen; but if the thickness be about $\frac{1}{20}$ of an inch, the spectrum will appear in detached portions, separated by broad and perfectly black intervals, the rays corresponding to these intervals being totally absorbed. If the thickness be diminished, the dark spaces will be partially illuminated; but if the thickness be increased, all the colors between the extreme *red* and *violet* will disappear.

SIR JOHN F. W. HERSCHEL conceived that the simplest hypothesis with regard to the extinction of a beam of homogeneous light, passing through a homogeneous medium is, that for every equal thickness of the medium traversed, an equal aliquot part of the number of rays which up to that time had escaped absorption, is extinguished. That is, if the $\frac{n}{m}^{th}$ part of the whole number of rays, which will be called c, of any homogeneous beam which enters a medium, be absorbed on passing through a thickness unity, there will remain,

$$c - \frac{n}{m} c = \frac{m - n}{m} c;$$

and if the $\frac{n}{m}^{th}$ part of this remainder be ab-

sorbed in passing through the next unit of thickness, there will remain

$$\frac{m-n}{m}\,c - \frac{n\,(m-n)}{m^2}\,c = \frac{\overline{m-n}^{\,2}}{m^2}\,c,$$

and through the third unit

$$\frac{\overline{m-n}^{\,2}}{m^2}\,c - \frac{n(m-n)^2}{m^3}\,c = \left(\frac{m-n}{m}\right)^3 c,$$

and through the whole thickness denoted by t units,

$$\left(\frac{m-n}{m}\right)^{t-1} c - \frac{n}{m}\left(\frac{m-n}{m}\right)^{t-1} = \left(\frac{m-n}{m}\right)^{t} c.$$

So that, calling c the number of equally illuminating rays of the extreme red in a beam of white light, c' that of the next degree of refrangibility, c'' that of the next, and so on, the beam of white light will, according to Sir J. H., be represented by

$$c + c' + c'' + c''' + \&c.$$

and the transmitted beam after traversing a thick-

ness t, by

$$c y^t + c' y'^t + c'' y''^t + \&c. \quad . \quad . \quad . \quad (89)$$

Wherein y represents the fraction $\frac{m - n}{m}$, which will depend upon the ray and the medium, and will, of course, vary from one term to another.

From this it is obvious, that total extinction will be impossible for any medium of finite thickness; but if the fraction y be small, then a moderate thickness, which enters as an exponent, will reduce the fraction to a value perfectly insensible.

Numerical values of the fractions y, y', y'', &c., may be called the *indices of transparency* of the different rays for the medium in question.

There is no body in nature perfectly transparent, though all are more or less so. Gold, one of the densest of metals, may be beaten out so thin as to admit the passage of light through it: the most opaque of bodies, charcoal, becomes one of the most beautifully transparent under a different state of aggregation, as in the diamond, "and all colored bodies, however deep their hues and however seemingly opaque, must necessarily be rendered visible by rays which have entered their surface; for if reflected at their surfaces, they would all appear white alike. Were the colors of bodies strictly superficial, no variation in their thickness could effect their hues; but so far is this from be-

ing the case, that all colored bodies, however intense their tint, become paler by diminution of thickness. Thus, the powders of all colored bodies, or the streak they leave when rubbed on substances harder than themselves, have much paler colors than the same bodies in mass."

95. By viewing the prismatic spectrum through media possessing different absorptive powers for the different rays, SIR DAVID BREWSTER has been able to detect red rays in the *blue* and *indigo* spaces ; *yellow* in the *red* and *blue,* and *blue* in the *red.* He has, moreover, been able to obtain white light from almost every part of the spectrum by absorbing the excess of those colors which predominate, and he hence infers that the solar spectrum consists of *three separate spectra of red, yellow and blue, all of equal length and occupying the same space ; the red having its maximum intensity about the middle of the red space, the maximum of the yellow being about the middle of the yellow, and that of the blue, between the blue and indigo.* The remaining colors of the spectrum, viz : *orange, green, indigo* and *violet,* he regards as resulting from the superposition of these three. Thus, let A C (fig. 75), represent the spectrum ; the ordinates of the curve A R C, the number of *red* rays at the corresponding points of the spectrum ; the ordinates of the curve A Y C, the same for the *yellow ;* and those of the curve A B C, the same for the *blue.* Now, at every point of the

spectrum there will be three ordinates, one of *red*, one of *yellow*, and one of *blue* ; and if one of these be selected so that portions may be laid off on the others bearing to this the relations which exist among the numbers expressive of the quantity of each of the three colors necessary to form white light, the remaining portions of these latter ordinates will express the excess of those colors which predominate. If these be *blue* and *yellow*, for instance, they will mark the *green* space in the spectrum, this latter color being known to result from the mixture of the former ; if red and yellow, the orange space ; if *red* and *blue*, the indigo or violet, according to the proportions.

Under this view of the constitution of the solar spectrum, *red*, *yellow* and *blue* are called primary colors, each possessing a refractive index varying in numerical value, between those corresponding to the extremes of the spectrum. Whenever, therefore, the index of refraction of any particular color is referred to, it must be understood as relating to that part of the spectrum marked by the middle of this color, and will belong alike to each of the three primary colors from whose union both white light and the particular color result.

96. When the spectrum is formed from light proceeding through a narrow slit, say about $\frac{1}{20}$ of an inch broad, the refracting edge of the prism being parallel to the length of the slit, it is found, on examination through a telescope, to be crossed

at right angles to its length, or parallel to the edge of the prism, by a series of dark parallel lines from one end to the other. They are about 600 in number, varying in distinctness, the largest subtending at the distance of the spectrum from the prism an angle of from 5″ to 10″, and the distances between them differing from each other. These lines are found in the spectra produced by all solid and liquid bodies, and whatever be the lengths of the spectra or colored spaces, they always occupy the same relative position within these spaces, provided the light coming either directly or indirectly from the sun be used. Similar lines are observed when the light of the fixed stars is employed, but they have been found to vary both in position and intensity.

The boundaries of the colored spaces of the spectrum being but ill defined, these fixed lines afford the means, which without them would be wanting, to determine with accuracy the refractive and dispersive powers of bodies.

Internal Reflection.

97. When an object is seen by reflection from a plate of glass, the faces of which are not parallel, it usually appears double. This is owing to the reflection which takes place at the second as well as first surface, and the image from the former

will be brighter as the obliquity or angle of incidence of the incident rays becomes greater. In this we have supposed the surrounding medium to be the atmosphere, between which and glass there is a great difference in refractive powers; but if the second surface of the glass be placed in contact with water, the brightness of the image from that surface will be diminished; if olive oil be substituted for the water, the diminution will be greater, and if the oil be replaced by pitch, softened by heat to produce accurate contact, the image will disappear. If, now, the contact be made with oil of cassia, the image will be restored; if with sulphur, the image will be brighter than with oil of cassia, and if with mercury or an amalgam, as in the common looking-glass, still brighter, much more so indeed than the image from the first surface.

The mean refractive indices of these substances are as follows:

Air,	1.0002
Water,	1.336
Olive Oil,	1.470
Pitch,	1.531 to 1.586
Plate glass,	1.514 to 1.583
Oil of Cassia,	1.641
Sulphur,	2.148

Taking the differences between the index of refraction for plate glass and those for the other substances of the table, and comparing these dif-

ferences with the foregoing statement, we are made acquainted with the fact, which is found to be general, viz : that when two media are in perfect contact, the intensity of the light reflected at their common surface will be less, the nearer their refractive indices approach to equality ; and when these are exactly equal, reflection will cease altogether.

98. Different substances, we have seen, have in general, different dispersive powers. Two media may, therefore, be placed in contact for each of which the same color as red, for example, may have the same index of refraction, while for the other elements of white light, the indices may be different ; when this is the case, according to what has just been said, the red would be wholly transmitted, while portions of the other colors would be reflected and impart to the image from the second surface the hue of the reflected beam ; and this would always occur, unless the media in contact possessed the same refractive and dispersive powers.

The Rainbow.

99. The rainbow is a circular arch, frequently seen in the heavens during a shower of rain, in a

direction from the observer opposite to that of the sun.

If A B C (fig. 76), be a section of a prism of water at right angles to its length by a vertical plane, and S r a beam of light proceeding from the sun; a part of the latter will be refracted at r, reflected at D, and again refracted at r', where the constituent elements of white light, which had been separated at r, will be made further divergent, the red taking the direction r' R, and the violet the direction r' V making, because of its greater refractive index, a greater angle than the red with the normal to the refracting surface at r'. To an observer whose eye is situated at E, the point r' will appear red, the other colors passing above the eye; and if the prism be depressed so as to occupy the position A′ B′ C′, making r'' V′, parallel to r' V, the point r'' would appear of a violet hue, the remaining colors from this position of the prism falling below the eye. In passing from the first to the second position, the prism would, therefore, present, in succession, all the colors of the solar spectrum. If now the faces of the prism be regarded as tangent planes to a spherical drop of water at the points where the two refractions and intermediate reflection take place, the prism may be abandoned and a drop of water substituted without altering the effect; and a number of these drops existing at the same time in the successive positions occupied by the prism

in its descent, would exhibit a series of colors in the order of the spectrum with the red at the top.

A line ES passing through the eye and the sun, is always parallel to the incident rays; and if the vertical plane revolve about this line, the drops will describe concentric circles, in crossing which, the rain in its descent will exhibit all the colors in the form of concentric arches having a common centre on the line joining the eye and the sun, produced in the front of the observer. When this line passes below the horizon, which will always be the case when the sun is above it, the bow will be less than a semi-circle; when it is in the horizon, the bow will be semi-circular.

To find the angle subtended at the eye by the radii of these colored arches, let A B D (fig. 77), be a section of a drop of rain through its centre; S A the incident, A D the refracted, D B the reflected, and B R the emergent rays. Call the angle C A m = the angle of incidence, φ, and the angle C A D = the angle of refraction, φ'; the angles subtended by the equal chords A D and D B, χ; and the angle A C B, θ. Then we we shall have

$$\theta = 2\pi - 2\chi;$$

and if there be two internal reflections (fig. 78), there will be three equal chords, in which case,

$$\theta = 2\pi - 3\chi;$$

and generally, for n internal reflections,

$$\theta = 2\pi - \overline{n+1} \cdot \chi \quad . \; . \; . \; . \quad (90)$$

but in each of the triangles whose bases are the equal chords, and common vertex the centre of the drop,

$$\chi = \pi - 2\varphi'$$

and this in equation (90) gives, on reduction,

$$\theta = 2(n+1)\varphi' - (n-1)\pi \quad . \quad (91).$$

Because the chords are all equal, the last angle of incidence C B D within the drop in fig. (77), or C B D′ in fig. (78), is equal to the angle of refraction C A D, and hence the angle of emergence C B m' is equal to the angle of incidence C A m.

The angle A O B in fig. (77), is the supplement of the total deviation of the emergent from the incident ray, and is equal to the angle B E F subtended by the radius of the bow; in fig. (78), it is the excess of total deviation above 180°.

Calling this angle δ, we shall have

$$\delta = \mp (2\varphi - \theta);$$

the upper sign referring to fig. (77), and the lower to fig. (78); replacing θ by its value in equation (91), the above reduces to

$$\delta = \mp (2\varphi - 2(n+1)\varphi' + \overline{n-1} \cdot \pi) \quad . \quad . \quad (92)$$

this, with equation

$$\sin\varphi = m \cdot \sin\varphi', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (93)$$

will enable us to determine the value of δ, when φ and m are given for any particular color.

For any value of φ assumed arbitrarily, δ will, in general, correspond to rays of the same color so much diffused as to produce little or no impression upon the eye; but if φ be taken such as to give δ a maximum or minimum, then will the rays of the color, corresponding to m, emerge parallel, or nearly so, for a small variation in the angle φ on either side of that from which this maximum or minimum value of δ results; hence, the

rays which enter the eye in this case will be sufficiently copious to produce the impression of color, and these are the rays that appertain to the rainbow. To find this value of φ, we have, from equation (92),

$$\frac{d\,\delta}{d\,\varphi} = \mp\left(2 - 2\,(n+1)\,\frac{d\,\varphi'}{d\,\varphi}\right)$$

but from equation (93) we obtain

$$\frac{d\,\varphi'}{d\,\varphi} = \frac{\cos\varphi}{m\cos\varphi'},$$

hence,

$$\frac{d\,\delta}{d\,\varphi} = 0 = \mp\left(2 - 2\,(n+1)\,\frac{\cos\varphi}{m\cos\varphi'}\right) \quad . \quad . \quad 94)$$

or

$$\frac{1}{n+1} = \frac{\cos\varphi}{m\cos\varphi'}.$$

Clearing the fraction, squaring both members, and adding

$$\sin^2\varphi = m^2\sin^2\varphi'$$

and reducing, we get

$$\cos\varphi = \sqrt{\frac{m^2 - 1}{n^2 + 2n}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

For one internal reflection, which answers to figure (77),

$$\cos\varphi = \sqrt{\frac{m^2 - 1}{3}};$$

and substituting in succession the value of m answering to the different colors for water, we shall have values for φ, and consequently for φ', equation (93), which substituted in equation (92), will give the angles subtended by the radii of the colored arches which make up what is called the *primary bow.*

For red, $m = 1{,}3333$, hence

$$\cos\varphi = .5092 = \cos 59^\circ\ 21'$$

$$\sin\varphi = \sqrt{1 - \cos^2\varphi} = .8603;$$

this last in equation (93), gives

$$\varphi' = 40^\circ\ 11'$$

and these values of φ and φ' in equation (92), give

$$\delta = -118^\circ.\,42' + 160^\circ.\,44' = 42^\circ.\,02'.$$

For the violet, $m = 1{,}3456$,

$$\begin{aligned}\cos\varphi &= \cdot 5199 = \cos 58^\circ.\,41\tfrac{1}{2}'\\ \sin\varphi &= \cdot 8543\\ \varphi' &= 39^\circ.\,25'\\ \delta' &= -117^\circ.\,23' + 157^\circ.\,40' = 40^\circ.\,17',\end{aligned}$$

hence, the width of the primary bow is

$$\delta - \delta' = 42^\circ.\,02' - 40^\circ.\,17' = 1^\circ.\,45'.$$

If there be two internal reflections, as in fig. (78), we shall, by making $n = 2$, find

$$\cos\varphi = \sqrt{\frac{m^2 - 1}{8}}$$

and obtain, by a process entirely similar, the elements of what is called the *secondary bow.*

For the red rays,

$$\delta = 50^\circ.57'$$

violet,

$$\delta' = 54^\circ.07'$$

and

$$\delta' - \delta = 3^\circ.10';$$

the value of δ' in the secondary, being greater than δ, the violet will occupy the outside, and the colors, therefore, be arranged in an order the reverse of that in the primary. Taking the difference between the values of δ in the primary and secondary bows, we will obtain the space between them, which is $50^\circ.57' - 42^\circ.02' = 8^\circ.55'$. The solar disk being about $32'$, the width of both bows must be increased by this quantity, the solution having been made upon the supposition that the light flows from a point. The primary is, therefore, $2^\circ.17'$ in width, and the secondary $3^\circ.42'$. The half of $32'$ being added to the radius of the red in the primary, will give $42^\circ.18'$, hence, if the sun be more than that height above the horizon, this bow cannot be seen. When more than $54^\circ.23'$, no part of the secondary will be visible.

By substituting in equation (95), 3 for n, we

might find the radii of a third bow, which would be found to encircle the sun at the distance of about 43°. 50′; but the proximity of the sun, together with the great loss of light arising from so many reflections, renders this bow so faint as to produce no impression; it is, therefore, never seen.

Differentiating equation (94), and reducing, it becomes

$$\frac{d^2\,\delta}{d\,\varphi^2} = \mp \left(- \frac{2\,(n+1)}{m} \cdot \frac{\sin\,(\varphi' - \varphi)}{\cos^2\,\varphi'} \right)$$

and since $\varphi' < \varphi$, the sign of $\sin\,(\varphi' - \varphi)$ will be negative, and hence, δ was a maximum for the primary and a minimum for the secondary. This explains the remarkable fact, (fig. 79), that the space between these bows always appears darker than any other part of the heavens in the vicinity of the bow, for no light twice refracted and once reflected can reach the eye till the drops arrive at the primary; and none which is twice refracted and twice reflected, can arrive at the eye after the drops pass the secondary; hence, while the drops are descending in the space between the bows, the light twice refracted with one and two intermediate reflections, will pass, the first above, and the second below or in front of the observer.

The same discussion will, of course, apply to the *lunar* rainbow which is sometimes seen.

100. Luminous and colored rings, called *halos*, are occasionally seen about the sun and moon; the most remarkable of these are generally at distances of about twenty-two and forty-five degrees from these luminaries, and may be accounted for upon the principle of unequal refrangibility of light. They most commonly occur in cold climates. It is known that ice crystalizes in minute prisms, having angles of 60°, and sometimes 90°; these floating in the atmosphere constitute a kind of mist, and having their axes in all possible directions, a number will always be found perpendicular to each plane passing through the sun or moon, and the eye of the observer. One of these planes is indicated in (fig. 80).

S m being a beam of light parallel to S E, drawn through the sun and the eye, and incident upon the face of a prism whose refracting angle is 90° or 60°, we shall have the value of δ, corresponding to a minimum from equation (8), by substituting the proper values of m for ice. The mean value being 1.31, we have

$$\sin \tfrac{1}{2}(\delta + 60) = 1.31 \,.\, \sin 30^\circ$$

$$\tfrac{1}{2}\delta = 40^\circ.\,55'.\,10'' - 30^\circ = 10^\circ\,55\,.\,10$$

$$\delta = 21^\circ.\,50'.\,20''$$

and

$$\sin \tfrac{1}{2}(\delta + 90) = 1.31 \, . \sin 45^\circ.$$

$$\tfrac{1}{2}\delta = 67^\circ . 52' - 45 = 22 . 52$$

$$\delta = 45^\circ . 44' ;$$

other phenomena of a similar nature will be noticed hereafter.

Interference of Light.

101. Let A B, A′ B′ (fig. 81), be two parallel linear radiants, such as may be formed by cylindrical lenses very near each other, from which proceed diverging rays of light. A portion of the rays from one of these radiants will cross a portion of those from the other, and if the whole be received upon a screen M N, parallel to the plane of the radiant, there will be an illuminated space consisting of a central portion due to the rays from both radiants, and two external portions, each enlightened by the rays from only one. So far as any property of light thus far noticed is concerned, we should expect to find the central portion of uniform brightness and of double the intensity of the others. Such, however, is not the case ; for when the radiants are very near each other, or the angle subtended by the distance between them at the

distance of the screen is very small, this space is found to be covered by *alternate stripes of light and total darkness,* the stripes being parallel to the radiants. These stripes are confined to the central space, and a bright one passes directly through the centre, the others alternating at distances on one side exactly equal to those which mark the alternations on the other; so that for each stripe on the right, for example, there will always be found a similar one at the same distance on the left. This will always be the case, no matter how the experiment be performed, provided the foregoing conditions be fulfilled; and hence it is obvious, that *the phenomenon must depend upon some property or affection of light itself.* If the light from one of the radiants be arrested by the interposition of some opaque substance, the stripes will disappear, and the illuminated space in the screen will become of one uniform brightness equal in intensity to that due to the other radiant; from which it follows, *that the rays from both radiants are necessary to produce the stripes.*

Considering, now, the paths of the crossing rays, it is manifest that the central bright stripe is equally distant from both radiants, and that the distances of any other stripe, whether bright or dark, from the radiants will be unequal, the difference depending upon the distance of the stripe from the centre. Hence it is inferred, "*that any ray divided along its length into intervals equal to these differences, must at the alternate points be*

somehow in a different state; such that if two rays cross at points where they are in the same condition, they conspire to form a BRIGHT *point; and if in different conditions, they neutralize and destroy each other, and leave a point of* DARKNESS."

This is called the INTERFERENCE of light.

If instead of receiving the light upon a screen, it be permitted to fall on a convex lens provided with a micrometer, by looking through the lens, the stripes will not only appear to greater advantage, but the distances between them may be accurately measured as well as the angular distance of the radiants; and hence the intervals along the rays at which the different states occur, may be determined.

To do this, it is in the first place obvious, that the points of the rays diverging from either radiant which are in the same condition, will be on the surfaces of concentric cylinders whose common axis is the radiant line; the surfaces about one radiant will intersect those about the other, and if the diverging rays be taken within very narrow limits, these surfaces may, within such limits, be regarded as planes whose intersections with a plane normal to the radiants will be right lines perpendicular to the rays. Thus A A′ (fig. 82), being the points of intersection of the radiants with the normal plane, and A B, A′ B, any two rays; then M N, M′ N′, $m\,n$, $m'\,n'$, which represent the intersections of the plane normal to the radiants A A′ with the cylindrical surfaces,

may be regarded as right lines perpendicular to their respective rays: hence, calling λ the distance between two consecutive cylinders, c the the distance O O′ on the screen or in the micrometer, between the central and next bright stripe in order, and $2\,\psi$ the angle subtended by the line joining the radiants, we get, considering B O equal to $\frac{1}{2}\lambda$,

$$\lambda = 2\,c\,.\tan\psi \quad . \quad . \quad . \quad . \quad . \quad (96)$$

hence,

$$c = \tfrac{1}{2}\lambda\,.\cot\psi.$$

and for the distance of any bright stripe from the centre

$$n\,c = c' = n\,\lambda\,.\frac{\cot\psi}{2} \quad . \quad . \quad . \quad . \quad (97)$$

from which it follows, λ being constant, that the stripes will be diminished as ψ is increased; a result confirmed by experience, for if the screen be made to approach the radiants, in which case ψ will increase, the stripes become narrower and extend over a much smaller space; again, if the screen radiants be made to separate, the same effect is observed till the stripes become so fine as not to be perceptible.

102. The law according to which any one stripe, the n^{th} for example, approaches the central bright stripe, when the screen is moved towards the radiants, is readily ascertained. This n^{th} stripe is formed by the union of two rays that differ in length by some multiple $n\lambda$ of λ, and both rays being diminished successively by λ, 2λ, 3λ, &c., the circles described with these diminished radii will intersect and determine the points between the screen and radiants at which this stripe will occur; and the locus of these intersections will be an hyperbola, since the difference in the lengths of the rays corresponding to its point is constant. Hence, the co-ordinates S P (fig. 83), of this curve referred to the line B B′, connecting the central bright stripe with the middle point between the radiants, regarded as an origin, will determine the law in question.

103. If the *linear* radiants become radiant *points*, the stripes upon the screen will assume the form of diverging *hyperbolas*.

104. When the radiant points (fig. 84), are taken at unequal distances from, and in the same line perpendicular to the screen, the stripes will take the shape of concentric circles around the point in the screen at which it is pierced by the line joining the radiants.

105. It follows from the existence of these similar and dissimilar points at equal intervals along any ray of light, that if two rays occupy the same position throughout their entire lengths, and the points of the one coincide with the similar points of the other, they will produce double the effect due to one of them separately; but if the dissimilar points coincide, they will destroy each other, and be rendered incapable of producing any effect whatever. If intermediate points should coincide, the effect will be intermediate between those just stated.

106. When the experiment of number (101) is made with the different kinds of homogeneous light, the stripes for the same position of the radiants and screen are found to vary, being broadest in red and narrowest in violet; hence the intervals along the ray at which the different states occur will not, equation (96), be the same for the different colors.

The following table, which is the result of very accurate experiments, exhibits the value of λ for the various colors of the spectrum.

Colors.	Length of λ in parts of an inch.	Number of λ in one inch.
Extreme red,	0.0000266	37640
Orange,	0.0000240	41610
Yellow,	0.0000227	44000
Green,	0.0000211	47460
Blue,	0.0000196	51110
Indigo,	0.0000185	54070
Extreme violet,	0.0000167	59750

and from which we may infer,

First. That when the experiment is performed in common light, the central white stripe results from the *superposition* of the various stripes of red, orange, and so on to violet; and because these elementary stripes are not of the same width, the central stripe will have an extremely small portion of color on its edges, red being on the outside on account of the greater width of this color.

Second. Leaving the central stripe and proceeding to the right and left, the amount of colors on the borders of the stripes will continually increase till they exhibit several colors in the order of the spectrum, the *red* being always at the greatest distance from the centre.

Third. The dark spaces a little beyond this, will be encroached upon by the further separation of the elementary colors till they disappear, and the whole system of stripes be lost.

107. Thus far the rays have been supposed to

proceed from the radiant to the screen through the atmosphere ; but if the experiment be performed in a medium of greater refractive power, the stripes will be reduced in width and crowded into a narrower space, and consequently the intervals along the rays at which the different states occur, will be shortened ; and this effect is found, according to very accurate experiment, to be exactly proportional to the index of refraction of the medium as referred to atmospheric air.

Returning to the experiment in air, if a very thin plate of glass be interposed in front of one of the radiants, and parallel to the plane of the radiants, the whole system of stripes will be shifted towards the side of the interposed glass. If an exactly similar plate be placed in front of the other radiant, and parallel to the first plate, the stripes will be restored to their original position. If one of the plates be slightly inclined, so as to cause the pencil passing through it to traverse a greater thickness, the stripes will all move towards that side, and by gradually increasing the inclination, the stripes will pass entirely out of the bright space illuminated by both radiants, and thus disappear.

Taking plates of any other medium, possessing a greater refractive power than glass, and of the same thickness as before, it is found that the effects just noticed will be increased, and in the direct ratio of the refractive indices.

In the shifting of the stripes, it is evident that the lengths of the rays producing the central one,

are rendered unequal, and that the differences as to length existing among the rays that form the other stripes, are not the same as before. It is now proposed to investigate this change.

For this purpose, let the rays from both radiants pass through a prism of any medium, as glass, having a very small refracting angle $= i$, (fig. 85), the first face being held parallel to the plane of the radiants. The thickness of the prism traversed by two interfering rays will be different; call this difference, which is $r\,n$ in the figure, d. Since the angle $2\,\psi$, made by the same rays, is very small, they will enter the first surface under very small angles of incidence, and both being refracted towards the perpendicular, their direction through the prism will be nearly normal to that surface; hence, denoting by b the distance $r\,n'$ between the rays measured on the second surface, we have

$$d = b \,.\, \sin i;$$

but under the above supposition, the angle of incidence at the second surface will be equal to i; and denoting the corresponding angle of refraction by φ, we also get

$$\sin \varphi = m \sin i;$$

with S, the new position of the central stripe, as a

centre, and radius S r, describe the arc $r\,r'$; $r'\,n'$ will be the difference of the interfering rays after leaving the prism. Calling this d', and because $r\,r'$ is very small, we have

$$d' = b \cdot \sin \varphi = b\,m \cdot \sin i = m\,d$$

hence,

$$d : d' :: 1 : m$$

that is, *the difference of lengths within the prism, is to that out of the prism, as unity is to the index of refraction.* But the number of intervals λ in d while in the prism, is to the number in an equal length in the air, as the index of refraction is to unity, and hence, *the number of intervals at which the different states occur will be the same in both rays, reckoning from the radiants to the point of interference; and for any other stripe, although the interfering rays will present a greater or less linear difference than before the prism was interposed, yet the difference in the number of intervals will be accurately the same.*

Whence it follows, that if two rays interfere and form any given stripe with a given difference of route, two other rays, having a greater or less difference will form the *same* stripe, provided one of the rays pass through a medium of such refractive

power as to make their difference, *as estimated by the number of intervals,* the same as before.

This fact of a greater or less number of intervals occurring within the same length of a ray in its passage through media of greater or less refractive power, is called, in the first case, the *retardation,* and in the second, the *acceleration of the intervals.*

The whole difference in the lengths of the interfering rays will be

$$d' - d = d(m - 1)$$

and returning to the parallel plate whose thickness will be denoted by t, we shall have by putting $n\lambda$, for $d' - d$,

$$n\lambda = t(m' - 1).$$

Substituting this value in equation (97), it gives

$$c' = t(m - 1)\frac{\cot\psi}{2} \quad . \quad . \quad (98)$$

Taking homogeneous light, and considering only

the extreme rays of the spectrum, viz: red and violet, denoted by r and v, we have

$$c'_r = t(m_r - 1) \cdot \frac{\cot \psi}{2}.$$

$$c'_v = t(m_v - 1) \cdot \frac{\cot \psi}{2},$$

and by subtraction

$$c'_r - c'_v = t(m_r - m_v) \cdot \frac{\cot \psi}{2} \quad . \quad (99).$$

When the first member exceeds the width of a stripe, the colors encroach upon the dark spaces, and the stripes will disappear. This is in perfect accordance with experience, for if the thin plate of glass be replaced by a thick one, the stripes will vanish.

Divergence of Light.

108. It has been stated (12), as a general fact, that the motion of light through a medium of homogeneous density, is rectilinear; to this, however, there is a remarkable exception. It is found

that when light passes near the edge or boundary of any body, though it continue in the same medium, it has a tendency to, and in a certain degree actually does, *diverge anew from this boundary as a new origin.* This is rendered evident by examining either the shadows of opaque bodies in diverging pencils of light, or the portions of these pencils transmitted through small openings and received upon a screen; in the first case the shadows will be less, and in the second the illuminated space will be greater than would result were the rays to pass in straight lines tangent to the bodies or edges of the openings. The pencil being accurately formed by a convex lens, and the dimensions of a small body placed in it being known, the fact of new divergence may be made matter of observation and distinct measurement.

Colored Fringes of Shadows and Apertures.

109. When an object is placed in a pencil, such as may be formed by admitting light through a very small aperture into a dark chamber, or by a convex lens, and the shadow of the object is received upon a screen, it is found to be bordered externally by bands, usually three in number at decreasing distances from each other, each band being made up of different colors. These bands are parallel to the outlines of the shadow, except

when the latter terminate in a salient angle, in which case, the bands curve around it; or when the outlines form a re-entering angle, when the bands will cross and run up to the shadow on each side.

In white light, the bands, reckoning from the shadow, are black, violet, deep blue, light blue, green, yellow and red; blue, yellow and red; pale blue, pale yellow and pale red.

In homogeneous light, the bands increase in number and are alternately dark and bright. In passing from one color to another, they vary in width, being broadest in red and narrowest in violet; and it is from the partial superposition of these and the remaining colors, that the different colors arise when the experiment is made with white light.

These bands are entirely independent of the nature and figure of the body whose shadow they surround, being the same when formed by a mass of platina or a bubble of air—by the back or edge of a razor.

The shadow being received upon a convex lens, behind which is placed a micrometer, the linear elements of the fringes may be measured to any desired degree of accuracy. These measurements show

1st. That the distances between the fringes and the shadow diminish as the lens approaches the body, and finally vanish, so that the fringes have their origin close to the edge of the body.

2d. That the locus of each fringe is an hyperboloid of revolution, terminating near the edge of the body.

3d. That, the distance of the lens from the body remaining the same, the fringes will be more dilated, as the body approaches the luminous point.

It is also found, that when the luminous point is increased so as to become an appreciable circle, the bands formed by the light proceeding from each of its points overlap and confuse each other, obliterating the colors, and forming a penumbra, which consists of a ring whose brightness varies from the edge of the shadow, where it is least, to its exterior boundary where it is greatest.

110. If the size of the body be much reduced in one direction, parallel to the screen or plane of the lens, the shadow will be found to consist of bright and dark stripes, parallel to the length of the body, a bright stripe occupying the centre. If the body be small and of a circular form, having its plane parallel to that of the screen, the shadow will be made up of a series of concentric bright and dark circles, having a bright spot in their centre.

As the body diminishes in size, the stripes diminish in number and increase in width, till all disappear but the central illumination. The reverse effect will arise either on increasing the size

of the body, or diminishing its distance from the screen.

111. If a portion of the pencil be transmitted through a small and well defined circular aperture and received upon a screen, concentric rings will also be produced; and if the transmitted portion be viewed through a convex lens, the hole will appear as a bright spot, encircled by rings of the most vivid colors, which undergo a great variety of changes, both as regards tint and linear dimensions, in varying the distance of the lens from the aperture, and that of the aperture from the radiant or luminous point.

When the light is transmitted through two very small apertures, very close together, rings corresponding to each will be formed as before, and in addition there will be found a number of straight parallel fringes between the centres of the circles, and at right angles to the line joining them; two other sets of parallel fringes will also be seen in the form of St. Andrew's cross proceeding from the space between the centres; and by multiplying the number of the apertures and varying their relative dimensions, a set of phenomena arise of exceeding brilliancy and beauty.

112. The colored fringes of shadows and small apertures, as well as all appearances referred to under this head, are explained upon the principle

of *interference;* the interference taking place between the rays of the original pencil that *diverge anew* from the edges of the body or aperture, and between these rays and those that experience no change in their course.

These phenomena were at one time called the *inflection* or *diffraction of light*, and were supposed to arise from some peculiar action exerted by the edges of bodies on the rays as they passed near them.

If the refractive power of the medium in which the experiments are performed be increased, the phenomena indicate a diminution in the intervals along the rays in the same ratio.

Double Refraction.

113. In treating of the transmitted portion of a beam of light incident upon any deviating surface, we have heretofore supposed all its rays to pursue a single course through the medium indicated by the constant ratio of the sine of incidence to that of refraction. This is not, however, always the case, there being a large class of bodies that possess the power of dividing the intromitted beam into two portions, each of which pursues, without further subdivision, a rectilinear path determined by its own peculiar law. This class embraces all crystallized media except those whose

primitive form is the *cube,* the *octohedron,* and the *rhomboidal dodecahedron;* all animal substances among whose particles there is a tendency to regular arrangement; and, in general, all solids in a state of unequal compression or dilatation.

The phenomenon presented by any medium of dividing a beam which enters it into two parts is called *double refraction.* One of the most remarkable bodies in this respect is *Iceland spar,* which is a carbonate of lime, and is found in crystals of various shapes, all of which may be reduced by cleavage to a *regular obtuse rhomboid,* this being its primitive form, or the form of its ultimate molecules. This rhomboid has six acute and two obtuse angles, and the diagonal A B (fig. 86), joining the latter, which is the shortest that can be drawn in the figure, is called the *axis of double refraction* or *optical axis* of the crystal; it is so called, because when the beam, on entering the crystal, moves along this line, no double refraction is observed. But this rhomb may be divided into an indefinite number of similar elementary rhombs each having an axis parallel to A B, it would, therefore, follow that whenever the refracted beam takes a direction *parallel* to A B, there should be no double refraction; and this is confirmed by observation. By an axis of double refraction, we are to understand, therefore, merely a *direction,* determined in the present case by joining the obtuse solid angles of the rhomb.

Whenever the beam takes a direction different

from the axis, it is found to be divided into two portions, one being refracted according to the ordinary law of the sines, the other according to some extraordinary law yet to be explained; the first portion is called the *ordinary ray*, the other, the *extraordinary ray*.

114. Any plane drawn through the crystal, parallel to the axis, is called a *plane of principal section*, and the peculiar property of such a plane is, that whenever it contains the ordinary ray, it will also contain the extraordinary ray; in other words, if a principal section, drawn through the point of incidence, coincide with the plane of incidence, it will contain both rays. Whenever these planes are inclined to each other, the extraordinary ray will not be in the same plane with the incident and ordinary rays, except in the case where the incident plane is perpendicular to the axis and consequently at right angles to every principal section possessed by the crystal, when the incident, ordinary and extraordinary rays will again be found in the same plane.

Let M N (fig. 87), represent a principal section of *Iceland spar*, normal to the upper face of the crystal, A B the optical axis, S P a beam of light, in the plane of principal section, incident at P. Now, if from the point P, the axis P X be drawn, the following are the facts determined by observation.

When the incident beam S P is normal to the

upper surface, the ordinary ray $P\,o$ will pass through without deviation, but the extraordinary ray will be *thrown from* the axis in the direction $P\,e$, making the angle $o\,P\,e$ upwards of six degrees. When the incident beam takes the position $S''\,P$, the ordinary ray $P\,o''$ will be refracted according to the ordinary law with a constant index of 1.654, and the extraordinary ray will take the direction $P\,e''$, making the angle $o''\,P\,e''$ greater than $o\,P\,e$, and this angle will reach its maximum limit when the extraordinary ray is at right angles to the axis. If the incident beam assume the position $S'\,P$, the ordinary ray will retain its constant index taking the direction $P\,o'$, and the extraordinary ray the direction $P\,e'$, making the angle $e'\,P\,o'$, less than $e\,P\,o$. Finally, when the incident beam $S'''\,P$ has a position such that the ordinary ray will be refracted along the axis, there will be no double refraction, or the angle $e'\,P\,o'$ will reduce to zero.

115. Had *rock crystal*, which occurs in the form of hexagonal prisms terminated with six sided pyramids been selected, what has been said of Iceland spar, would equally apply to it, except that the extraordinary ray is always found between the ordinary ray and the axis passing through the point of incidence, as if *drawn towards* the latter; and this circumstance has given rise to a division of doubly refracting substances into two

classes distinguished by their axes, which are said to be *positive* when the extraordinary ray is between the ordinary ray and the axis, as in the case of rock crystal; and *negative*, when the positions of these rays are reversed with respect to the axis, as in Iceland spar.

116. If we now suppose, for the purpose of explanation, the angle of incidence, S P P′ (fig. 88), to remain the same, the ordinary ray P O will be stationary, no matter what the position of the optical axis, since they are independent of each other. Not so, however, with the extraordinary ray P e; this will, from what has been said, assume, in the negative crystal, some direction as P e', when the axis takes the position P X′; and since the index of refraction for the extraordinary ray is the ratio of the sines of the angles P′ P S and P″ P e', this index is variable and less than that for the ordinary ray in every position of the axis except P O, when they will be equal; it will, of course, have its minimum value when the extraordinary ray and axis are at right angles. The reverse will be the case in crystals with a positive axis.

117. Having ascertained by experiment the value of the ordinary index, which will be represented by m_o, and the maximum or minimum value of the extraordinary index, according as the crystal has a positive or negative axis, which will

be represented by (m_e), it has been determined from theoretical considerations, and confirmed by experiment, that all values of the extraordinary index between m_o and m_e may be found by the following law.

Let an ellipsoid of revolution (fig. 89), be conceived, having its centre C at the point of incidence, its axis of revolution coincident with the optical axis of the crystal, and its polar to its equatorial radius in the inverse ratio of the minimum and maximum values of the extraordinary index of refraction; then, in all positions of the extraordinary ray, its index is equal to the reciprocal of its length contained between the centre and surface of the ellipsoid.

The equation of a section of this surface through the axis, referred to the centre, is

$$A^2 y^2 + B^2 x^2 = A^2 B^2,$$

and calling r, the length of the extraordinary ray between the centre and the surface, and θ, its angle of inclination with the optical axis, it reduces to

$$r = \frac{AB}{\sqrt{A^2 \sin^2\theta + B^2 \cos^2\theta}} = \frac{AB}{\sqrt{B^2 + (A^2 - B^2)\sin^2\theta}};$$

denoting by $m_{e'}$, the value of the extraordinary

index sought, we have

$$m_e = \frac{1}{r} = \sqrt{\frac{1}{A^2} + \left(\frac{1}{B^2} - \frac{1}{A^2}\right)\sin^2\theta} \quad . \; . \; . \; (100)$$

in which

$$A = \frac{1}{m_o},$$

$$B = \frac{1}{m_e}.$$

It is obvious that the coefficient of $\sin^2\theta$ is positive or negative according as the axis is positive or negative; hence, the coefficient of $\sin^2\theta$ determines the nature of the crystal.

118. To determine the value of m_o and m_e, in any particular instance, it is in the first place known that the index of the extraordinary ray will be constant and equal to its maximum or minimum value, according to the nature of the body, when refracted in a plane at right angles to the optical axis; it is only necessary, therefore, to convert the crystal, by grinding, into a prism whose refracting faces shall be parallel to the axis, when both the ordinary and extraordinary index may be ascertained by the method explained in (25). To distinguish between the rays, it will, in general, be sufficient to move the prism so as to give the

plane of incidence a slight inclination to its length, as in that case the extraordinary ray will be thrown out of this plane, and thus become known.

In Iceland spar

$$m_o = 1.6543,$$

$$m_e = 1.4833;$$

hence,

$$A = 0.60449,$$

$$B = 0.67417;$$

the ellipsoid is, therefore, oblate; and the coefficient of $\sin^2 \theta$, negative. Tourmaline, beryl, emerald, apatite, &c., also belong to this class. Quartz, ice, zercon, oxide of tin, &c., give the coefficient of $\sin^2 \theta$ positive; they are, therefore, of the positive class, and the ellipsoid is prolate.

119. Among doubly refracting crystals there are very many that possess two axes of double refraction, but in all such cases it has been ascertained that there is, in fact, no ordinary ray.

Polarization of Light.

120. When a beam of light is incident upon any deviating surface, it has been before remarked

that a portion is always reflected and another transmitted; and the relative intensity of these will be constant so long as the surface and angle of incidence remain the same, no matter to which *side* of the beam the deviating surface be presented, provided, the light be in the state in which it comes from the *sun* or any *self luminous* body.

But with light that *has already undergone some reflection, refraction, or other action of material bodies,* this uniformity of result will not obtain. Such light is found to have acquired different properties on *different sides,* for the intensity of the reflected and transmitted portions are found materially to depend on the *side* of the beam to which the deviating surface is offered. A beam or ray, distinguished by this, and other circumstances to be noticed hereafter, is said to be *polarized.*

Polarization by Reflection.

121. The intensity of the reflected portion of a beam of light, is found to be greater in proportion as the refractive index of the medium, and angle of incidence are greater. It is, moreover, ascertained that when reflection from any *transparent* medium takes place under a certain angle of incidence, called the *polarizing angle,* the reflected beam loses almost entirely the power of being again reflected when the reflector is presented in a particular manner.

M N, and M′ N′ (fig. 90), representing two plates of glass, mounted upon swing frames, attached to two tubes A and B, which move freely one within the other about a common axis, let the beam S D, from any self luminous body, be received upon the first under an angle of incidence equal to 56° ; reflection will take place according to the ordinary law in a plane normal to the reflecting surface ; and if the reflected beam D D′, which is supposed to coincide with the common axis of the tubes, be incident upon the second reflector under the same angle of incidence, *the reflector being perpendicular to the plane of first reflection*, it will be again reflected in the same manner as before.

But if the tube B be turned about its axis, the tube A being at rest, the angle of incidence on the glass M′ N′ will remain unchanged, yet the portion reflected from it will become less and less, till the tube B has been turned through an angle equal to 90°, as indicated by the graduated circle C, on the tube A, when the beam will almost totally disappear, or cease to be reflected. Continuing to turn the tube B, the reflection from M′ N′ will increase till the angle is equal to 180°, when the plane of first reflection will be again perpendicular to M′ N′, and the whole beam will be reflected ; beyond this, reflection will diminish till the angle becomes 270°, when the beam will be again lost ; after passing this point, the lost beam will be

gradually restored, till the tube is revolved through 360°, when the restoration will be complete.

It thus appears that a beam of light reflected from a plate of glass under an angle of incidence equal to 56°, immediately acquires *opposite* properties, with respect to reflection, on sides distant from each other equal to 90°, measuring around the beam ; and the *same* property at distances of 180°.

We have supposed the angle of incidence 56°, if it were less or greater than this, similar effects would be observed, though less in degree ; or, in other words, the beam would appear but *partially* polarized, the palarizing effect decreasing as the angle of incidence recedes from that of polarization, being nothing at the incidence of zero and 90°.

The plate M′ N′ is called the *analyzer;* the plane of first reflection is called the *plane of polarization,* and the beam is said to be polarized in this plane. The position of this plane in any polarized beam may readily be ascertained by the total reflection which takes place from the analyzer when the latter is perpendicular to it. Starting from this position of the analyzer with respect to the plane of polarization, and calling *d,* the angle between the plane of polarization and that of second incidence, which is equal to the angle through which the analyzer has at any time been turned about the first reflected or polarized beam ; A, the intensity of this beam, and I, the variable

intensity of that reflected from the analyzer in its various positions, it has been conceived, on careful investigation, that in uncrystalized media the formula

$$I = A \cos^2 d \quad . \quad . \quad . \quad . \quad . \quad (101)$$

will express the law according to which a polarized beam will be reflected from the analyzer when the angle of incidence is equal to that of polarization.

According to this law, if we conceive a common beam, as it emanates from any self-luminous body, to be composed of two beams polarized in planes at right angles to each other, we should have, calling I and I′ the intensity of the reflection in the first and second respectively,

$$I + I' = A \cos^2 a + A \cdot \cos^2 (90^\circ - a) = A$$

or the intensity of the reflected beam will be the same on whatever side of the incident beam the analyzer is presented.

118. What has been said of the effects of glass on light is equally true of other transparent media, except that the polarizing angle, which is constant for the same substance, differs for different bodies,

Sir David Brewster discovered, from very numerous observations, that *the tangent of the maximum polarizing angle is always equal to the refractive index of the reflecting medium taken in reference to that in which the ray is reflected:* thus, calling the relative index m, and the polarizing angle φ, we shall have,

$$\tan \varphi = m. \qquad \qquad (102)$$

Example. Let it be required to find the polarizing angle when light is moving in water and reflected from glass. The refractive index for water and glass are 1,336 and 1,525, respectively, hence,

$$m = \frac{1.525}{1.336} = 1.1415 = \tan \varphi$$

or

$$\varphi = 48°.47'.$$

If the refractive indices of the media were equal, we should have

$$m = 1$$

and

$$\varphi = 45°.$$

The following are the values of φ, for the different substances named, the ray being reflected in air.

Water,	53°. 11′
Crown glass,	56°. 55′
Plate glass,	57°. 45
Oil of Cassia,	58°. 39
Diamond,	68°. 6′

123. It is obvious that according to the law expressed by equation (102), there can be no such thing as perfect polarization by reflection in a beam of white light, since the refractive index is not the same for the different colors; and hence there can never be total absence of light at the analyzer; but *a certain tint will be reflected, whose intensity will depend upon the dispersive power of the medium.* For bodies of very high refractive powers, which are also, in general, highly dispersive, we must, therefore, understand by the polarizing angle, that angle of incidence at which the reflected beam approaches nearest to perfect polarization. This angle being ascertained for opaque bodies by experiment, the relation expressed by equation (102), furnishes the means of ascertaining their refractive indices. Thus, the maximum polarizing angle for steel is a little over 71°, the natural tangent of which is 2.85, which is, therefore, according to the law, its refractive index: the polarizing angle for mercury

is about 76°. 30′, and its index, consequently, 4.16.

With metallic substances, no complete polarization takes place.

124. We have spoken, thus far, only of the action at the *first* surface of the glass plate; it is found that the light reflected at the *second* surface is as perfectly polarized as that reflected at the first, and in the same plane, when the faces of the plate are parallel. This is a consequence of the same law for, (fig. 91),

$$m = \tan \varphi = \frac{\sin \varphi}{\cos \varphi} = \frac{\sin \varphi}{\sin \varphi'}$$

hence,

$$\cos \varphi = \sin \varphi'$$

or φ' is the complement of φ, and the first reflected beam is perpendicular to the first refracted.

Moreover,

$$\frac{1}{m} = \frac{1}{\tan \varphi} = \cot \varphi = \tan \varphi'$$

but $\frac{1}{m}$ is the index of the ray passing out of the glass; hence φ' is the maximum polarizing angle for the second surface.

If a series of parallel plates be employed in the form of a pile, the rays reflected from the second surface coming off polarized in the same plane, a very intense polarized beam may be obtained. This beam will, however, never contain more than half the original incident beam, no matter how great the number of plates employed.

125. Although a beam of light is but imperfectly polarized when reflected once at an angle differing from that of polarization, yet by repeating the reflections a sufficient number of times the polarzaition may be completed; and in doing this it is not necessary that the reflections take place at the same angle of incidence, but some may be above and some below the polarizing angle. In general, the number of reflections will increase as the angle of incidence recedes from that of polarization on either side.

Polarization by Refraction.

126. When a beam of light S P (fig. 92), is incident upon several plates of glass, laid one above the other, the transmitted portion P′ S′, will be found partially polarized *in a plane at right angles to that of refraction*, and the degree of polarization will be greater as the number of plates is increased, so that if the number be considerable, the

ray may be wholly polarized. The polarizing effect of the plates will increase with the refractive indices of the media of which they are composed.

It is ascertained that at all angles of incidence upon a plate of glass, the reflected and transmitted portions of a beam contain equal quantities of polarized light, the planes of polarization being at right angles to each other.

Polarization by Absorption.

127. A plate of *Tourmaline,* about $\frac{1}{20}$ of an inch thick, cut parallel to the axis, possesses the property of intercepting a part of a beam of light and transmitting the other perfectly polarized in a plane at right angles to the axis of the crystal. If light previously polarized, be incident upon the plate with its plane of polarization perpendicular to the axis, it will be wholly transmitted; but if parallel, it will be wholly absorbed or intercepted. This is another property by which polarized light may be distinguished. Hence, two plates of tourmaline form a most convenient apparatus for experimenting with polarized light when so arranged as to be capable of turning about a common axis, the one being used to polarize light, the other to analyze it. Plates of agate and some varieties of quartz possess similar properties.

Polarization by Double Refraction.

128. It was stated (114), that a beam of light, S P (fig. 93), incident perpendicularly on the upper surface of a crystal of Iceland spar, will be divided into two parts, one of which P *e* will be refracted according to the extraordinary law, and the other P *o* will pass through without deviation. These beams, which will leave the crystal in the plane of principal section, normal to its upper face, in parallel directions, are found to be polarized in planes at right angles to each other, that of the ordinary ray being coincident with, and that of the extraordinary ray perpendicular to, the plane of principal section.

If these rays be received upon the upper face of a second crystal whose optical axis is parallel to that of the first, the beams $e\,e'$ and $o\,o'$ it is found, will not be again divided, but the whole of the latter will undergo ordinary, while the whole of the former will experience extraordinary refraction; and if the crystals be of equal thickness, the separation or distance $e''\,o''$, will be double $e\,o$. If either or both of the beams $e\,e'$, $o\,o'$ had been polarized by reflection, refraction or absorption, the action of the second prism would be precisely the same; this is, therefore, another characteristic property of polarized light, viz. that it will not undergo double refraction when its plane of pola-

rization is either *parallel* or *perpendicular* to the plane of principal section; being in the former case wholly refracted according to the *ordinary*, and in the latter according to the *extraordinary* law. The reverse would have been the case if the crystal, like quartz, had possessed a positive axis.

129. When the crystals M′ N′ is turned around on its base so that the principal sections of the crystals, which are normal to the upper surfaces, make an angle with each other, each of the beams $o\,o'$ and $e\,e'$ will be again divided into an ordinary and extraordinary beam, whose relative intensities will depend upon the inclination of the principal sections to each other. To avoid complication, let us suppose the beam P e to be arrested by sticking a piece of wafer to the lower surface of the first crystal at e, then will the intensities of the portions into which the beam $o\,o'$ is divided by the second crystal be expressed by the formulas

$$\left.\begin{aligned} O_o &= A\,.\cos^2\alpha \\ O_e &= A\,.\sin^2\alpha \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (103)$$

Wherein A, represents the intensity of the beam $o\,o'$; α, the angle made by the principal sections of the crystals; O_o the intensity of the ordinarily

refracted portion; and O_e that of the portion refracted according to the extraordinary law.

Removing the wafer from e, and calling E_e and E_o the intensities of the extraordinary and ordinary beams into which P_e is separated by the second crystal, and B its intensity on leaving the first crystal, we shall, in like manner, have

$$\left.\begin{aligned} E_e &= B \cdot \cos^2 \alpha, \\ E_o &= B \cdot \sin^2 \alpha; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (104)$$

taking the sum of the four emergent beams, there will result,

$$O_o + O_e + E_e + E_o = A + B.$$

The beams O_o and O_e, in equations (103), are always found to be polarized, the former in the plane of principal section of the second crystal, the latter in a plane at right angles to it; and the same remark being applicable to E_o and E_e, in equations (104), it follows that the planes of polarization of O_o and E_o will be parallel to each other, as also those of O_e and E_e.

Interference of Polarized Light.

130. Let $a\,b\,c$ (fig. 94), represent a thin plate

of Iceland spar, made by sections perpendicular to the optical axis. A pencil of common light being incident upon it in such direction that its central ray shall coincide with the axis of double refraction, this ray will pass through without being divided; but not so with the other rays. These, taking a direction different from that of the axis, will undergo double refraction, and the index of refraction for any extraordinary ray, depending upon the value of θ, in equation (100), will vary as the ray is taken at a greater or less distance from the axis. The acceleration of the intervals at which the different states occur on these rays in passing through the plate, will, therefore, (107), be variable, while that for the ordinary rays will be constant. In a *positive* crystal, the acceleration for the ordinary rays will be least, and in the *negative* greatest.

The two classes of rays (fig. 95), being superposed, or nearly so, on leaving the plate, will, in consequence of this difference of acceleration, be in a condition to interfere; and since the rays are symmetrically disposed about the axis, we should expect, on applying the eye to the opposite side of the plate, to find that line surrounded by a number of concentric circular rings or fringes, whose appearance would depend upon the difference in the accelerations above referred to, and the thickness of the plate within which the cause of that difference is permitted to act. No such fringes, however, are observed under the circumstances men-

tioned, and the reasons are found in the following laws relating to the interference of polarized light, which are due to M. M. Fresnel and Arago.

1*st*. *Two rays polarized in the* SAME *plane, interfere with each other just as natural light.*

2*d*. *Two rays polarized in planes at* RIGHT ANGLES *to each other, will not interfere under the same circumstances that cause the interference of natural rays.*

3*d*. *Two rays primitively polarized in* OPPOSITE PLANES, *(i. e. at right angles to each other), may afterwards be reduced to the same plane without acquiring the power of interfering with each other.*

4*th*. *Rays polarized in opposite planes, and then reduced to the same plane, acquire the power of interfering with each other, provided they belong to a beam* THE WHOLE OF WHICH *had been previously polarized in* ONE AND THE SAME PLANE.

5*th*. *In the interference of polarized rays produced by double refraction, the place of the colored fringes, is not alone produced by the difference of route as estimated by the number of intervals, but in certain circumstances a difference of* HALF AN INTERVAL *must be allowed for.*

The first and second of these laws are proved by the following experiment.

M N (fig. 96), represents a thin plate or lamina of sulphate of lime, upon which are incident the rays of natural light proceeding from two narrow slits L, R, very close together, in a thin sheet of

copper. As the sulphate possesses the power of double refraction, each pencil, (such we may consider the collection of rays proceeding from the slits), will be divided into two, an ordinary and extraordinary one, which will be of equal intensity, and those marked *e*, will be polarized oppositely to those marked *o*. Four combinations may, therefore, be formed, viz.

1st. R_o, may interfere with L_o;

2d. R_e, " " L_e;

3d. R_o, " " L_e;

4th. R_e, " " L_o.

Of these R_o and L_o are similarly polarized, and they have been equally accelerated; therefore, if they interfere, they will give rise to a system of fringes corresponding to the middle of the space between two planes through the slits and perpendicular to the sheet of copper. The same may be said of L_e and R_e, and hence these two sets of fringes will be superposed and appear as one of double intensity. A plane midway between those through the slits, will be called the *axis plane*.

If R_o and L_e, which are oppositely polarized, interfere, the fringes thence arising must, (107), be thrown to that side of the axis plane on which the

intervals are most accelerated, the distance from this plane depending upon the difference of acceleration and the thickness of the lamina. In like manner, on the supposition that R_e and L_o interfere, their fringes would be thrown to the opposite side of the axis plane. Only one set of fringes is, however, seen, viz: those corresponding to the axis plane; therefore, *the rays oppositely polarized do not interfere.*

But if the plate of sulphate be cut in two, at its intersection with the axis plane, and one half be turned in its own plane through 90°, R_e and L_o, and L_e and R_o, will become similarly polarized, while the reverse will be the case with R_o and L_o, L_e and R_e; and it is, accordingly, found that the central fringes will disappear and the lateral fringes start into existence; the half lamina being still further revolved, the latter fringes disappear and the others return.

The truth of the 3d law may be thus shown. Take two perfectly similar plates of tourmaline, and apply one before each slit in the copper, and in such manner (123), that the two transmitted pencils, denoted by L and R, shall be oppositely polarized. These being received upon a doubly refracting crystal, whose plane of principal section is inclined 45° to their planes of polarization, they will each be divided into two of equal intensity, giving two pairs polarized in the same plane. We ought, therefore, expect to find two sets of

fringes, one produced by the interference of the ordinary rays from R and L, the other by that of the extraordinary rays from R and L, yet no fringes are seen.

The following experiment is mentioned in support of the 4th and 5th laws. A lamina of sulphate of lime is presented perpendicularly to a *polarized pencil*, with its plane of principal section making an angle of 45° with the plane of polarization, and immediately behind this is a thin plate of copper pierced with two very narrow slits, denoted by R and L, near each other, from each of which, emerge two equal pencils,

$$\left.\begin{array}{l} R\,o, \\ R\,e, \end{array}\right\}$$

$$\left.\begin{array}{l} L\,o, \\ L\,e, \end{array}\right\}$$

oppositely polarized, viz: in planes making angles of + 45° and − 45° with the plane of primitive polarization. A rhomboid of Iceland spar is now interposed with its principal section parallel to the primitive plane of polarization, and therefore, making an angle with that of the lamina of 45°. Each of the four pencils is thus divided into

two equal pencils, an ordinary and extraordinary, giving rise to eight pencils in all, viz.

$\overbrace{Roo, Reo}$; $\overbrace{Loo, Leo}$; $\overbrace{Roe, Ree}$; $\overbrace{Loe, Lee}$.

Any corresponding rays of the pencils *Ro* and *Re*, after quitting the lamina, are parallel, and, when the thickness of the lamina is small, may be regarded as superposed; but they have, within the lamina, been differently accelerated by a space which may be called d, so that if χ represent that portion of λ which determines the point or particular phase in any interval at which any one ray of R*o* leaves the crystal, $\chi + d$ will be that for the corresponding ray of R*e*. The same may be said of L*o* and L*e*. Moreover, the rays of either of these two pairs of pencils are oppositely polarized, viz: in planes inclined $+45^\circ$ and -45° to the plane of primitive polarization, and all the circumstances of their phase and position may be thus stated:

Ray	Phase	Inclination of Plane of polr. to P. P.
R*o*	χ	$+45^\circ$.
R*e*	$\chi+d$	-45°.
L*o*	χ	$+45$
L*e*	$\chi+d$	-45.

Calling δ, the difference of phase of the ordinary and extraordinary rays, into which each of the above pencils is divided by the Iceland spar, the statement becomes,

A			*B*		
Rays	Phase	Incl. plane.	Rays	Phase	Incl. plane,&c.
R *o o*	χ	0°	R *o e*	$\chi+\delta$	90°.
R *e o*	$\chi+d$	0°	R *e e*	$\chi+d+\delta$	90°.
L *o o*	χ	0°	L *o e*	$\chi+\delta$	90°.
L *e o*	$\chi+d$	0°	L *e e*	$\chi+d+\delta$	90°.

If the rhomb of Iceland spar be of considerable thickness, the group *A* will be so far removed from the group *B* as to prevent the possibility of their mixing with each other ; the rhomb is, in the first place, so taken.

Considering the first group *A*, R *o o* may combine with L *o o*, and since they have been equally accelerated, having no difference of phase, they should, if they interfere, produce fringes immedidiately about the axis plane ; and the same is true of R *e o* and L *e o*, and hence this set of fringes should be of double intensity.

Next *R o o* and *R e o*, may combine, but being differently accelerated, their fringes would be thrown to one side. R *e o* and L *o o*, may also combine and produce fringes as far removed to the opposite side of the axis plane.

Thus in the ordinary group A, there should be *three* sets of fringes, and the same number in the extraordinary group B for the same reason. Now these fringes are actually found to exist as may be seen by viewing the arrangement above described through a lens. It is evident that the *lateral* sets of fringes are formed by the interference of those rays that had been oppositely polarized by the sulphate of lime, and afterwards reduced to the same plane, and hence the truth of the 4th law.

If now, the Iceland spar be diminished in thickness till the group B be superposed upon the group A, we should expect to find but *three* instead of six sets of fringes, the middle one being still the brightest. But the fact is, but one set is seen, the lateral fringes disappearing entirely; which shows, that the colors resulting from the interference of the ordinary rays refracted by the rhomboid are complimentary to those produced by the interference of the extraordinary rays, and, therefore, that half an interval or space, necessary to produce this effect, must be gained or lost when a ray passes from the ordinary to the extraordinary class.

131. In the case of a diverging pencil of natural light transmitted along the axis of a doubly refracting crystal, with which we commenced this subject, the rays are primitively polarized in

planes at right angles to each other, and hence law 2d, the absence of the rings.

Besides, the axis of the pencil being coincident with that of double refraction, every ray, both ordinary and extraordinary, will lie in a plane of principal section; the planes of polarization of the former will, therefore, radiate in all directions from the axis, while those of the latter will be perpendicular to these, and may be regarded as so many tangent planes to the cones formed by rays at the same distance from the axis. The analyzer A (fig. 97), being presented under the polarizing angle to the emergent pencil, all the ordinary rays in the plane of principal section which is coincident with that of reflection D C will be reflected, but the extraordinary rays of this plane will disappear; and all the ordinary rays in the principal section A B, at right angles to this, will disappear, while its extraordinary rays will be totally reflected, from the principal sections intermediate between these, there will be partial reflection of both classes of rays, the number of each being equal at the azimuth of 45°; the ordinary rays being in excess when the azimuth is less than this, the extraordinary when greater. By reflection from the analyzer the planes of polarization are brought together, and the reflected pencil will be polarized in the same plane; but its rays having been originally polarized in *opposite planes*, there will still, according to law 3d, be a defalcation of the rings.

If instead of a pencil of natural light, one wholly polarized in the same plane be transmitted through the plate D (fig. 98), and the analyzer be applied under the polarizing angle, and in such position that the plane of incidence make an angle of 90° with that of polarization, the rings, exhibiting the most beautiful colors intersected by a black cross (fig. 99), will appear. If the crystal remain stationary and the analyzer be turned about the axis of the polarized pencil, the black cross will appear to dilate and grow fainter till the plane of incidence has been turned through an azimuth of 45°, beyond which, the cross, continuing to increase in brightness, will contract, and, finally, when the azimuth becomes 90°, or the plane of primitive polarization coincides with that of incidence, obtain its former dimensions and be perfectly white as in figure 100. Continuing the motion of the analyzer, the reverse appearance will take place and the black cross be restored when the azimuth is equal to 180°.

The explanation of all this seems quite obvious. The original pencil being wholly polarized in one plane, those of its rays which pass through the plate in the principal section which coincides with this plane, will all undergo ordinary, while those contained in the principal section at right angles to this, will undergo extraordinary refraction, without having their planes of polarization in the least changed. In the first position of the analyzer not one of these rays will be reflected, and hence the

black cross. When the plane of reflection is brought, by turning the analyzer, to coincide with the plane of polarization of these rays, which is that of the original pencil, they will all be reflected, and hence the *white cross.* The rays that pass through in principal sections other than those mentioned, will be doubly refracted agreeably to the law expressed in equation (103), α representing the angle made by any principal section with the plane of primitive polarization, and each principal section will contain a set of ordinary and extraordinary rays whose planes of polarization will be at right angles to each other; these falling upon the analyzer, in its first position, with their planes inclined to that of reflection between the limits of total suppression and reflection, will undergo partial reflection, and again be reduced to the same plane; and because one set, in this case the ordinary, was accelerated more than the other, and, having once belonged to a pencil or beam wholly polarized in the same plane, they will, according to law 4th, interfere and produce the fringes.

For the reflecting analyzer, substitute one of double refraction, say of Iceland spar, with its plane of principal section at right angles to that of primitive polarization. Two sets of fringes will now be seen, each exhibiting colors complimentry to the other, the ordinary image showing the black cross and extraordinary image the white.

The crosses are, as before, easily accounted for. In the position assumed for the analyzer, all

rays not divided by the first crystal, being those contained in two planes perpendicular to each other, one coinciding with the primitive plane of polarization, will undergo extraordinary refraction, and will hence be seen in the extraordinary, but not in the ordinary image.

To explain the formation of the colored rings, assume any ray of the original beam, say one contained in a principal section M N (fig. 101), inclined 45° to the plane of primitive polarization; and let A represent the system of semi-intervals of this ray before entering the crystal M N. In passing this crystal, the ray will be divided into two equal portions, and $d + \frac{\lambda}{2}$, will, agreeably to 5th law, be the difference of phase with which the rays will emerge, and B will represent the semi-intervals after the first refraction; the rays will also, on account of the small thickness of the crystal, be superposed, or nearly so, and be in a condition to interfere if their planes of polarization were the same. On reaching the analyzer, each ray is again divided into two equal parts, the ordinary rays of each preserving, with reference to each other, the same phase C, as at B; but the extraordinary portion of the first ordinary ray, gaining or losing by law 5th, half an interval on the extraordinary portion of the first extraordinary ray, these rays will have the analyzer in the condition represented at C′. The rays of each class passing the analyzer parallel, will emerge in the

same state, be superposed, and produce, by their interference, complementary colors.

Circular Polarization.

As a general fact, it has been observed, that when a polarized ray is passed through a doubly refracting crystal in the direction of its optical axis, it undergoes no change, the analyzer producing the same effect after as before transmission on being offered to the same side of the ray.

To this, however, there are some remarkable exceptions, one of the most conspicuous of which is exhibited by rock crystal, or quartz. When the analyzer is in the position affording no reflection, the interposition of a crystal of this substance will restore a portion of light, and to cause it to disappear, the analyzer must be turned through *a certain angle* about the ray, the magnitude of this angle depending upon the thickness of the interposed quartz. As the thickness of the mineral is increased, the rotation of the analyzer must be continued. The plane of polarization is thus twisted into a surface of double curvature resembling the turns of an auger, or the surface generated by the rotation of one right line about another perpendicular to it, at the same time that it has motion of translation along this second line. This is called *circular polarization*.

It has been found that for a given thickness, the

arc of rotation necessary to bring the analyzer into the position of evanescence is different for the differently colored rays, this arc being given by the formula

$$r = \frac{k\,t}{\lambda^2},$$

in which r represents the arc, k a constant, t the thickness, and λ the *interval* for the particular color.

It has also been ascertained, that for some specimens of quartz, it was necessary to turn the analyzer in one direction, while for others, in an opposite direction, and that a singular connection exists between this property and the right or left handed direction in which certain small faces of the crystal lean around the summit of the variety called plagiedral quartz.

Other bodies, besides quartz, possess the property of circular polarization, but in different degrees; and if two of these bodies be interposed, the arc of rotation is that due to the sum or difference of their thickness, according as they possess the same or opposite kinds of circular polarization. Or more generally,

$$\text{R T} = r\,t + r'\,t' + r''\,t'' + \&c.,$$

in which R is the rotation due to the combination; T its entire thickness; r, r', &c., and t, t', &c., the corresponding quantities answering to the several individuals of the combination; the products entering the expression with the same or different signs, according as the different media tend to turn the plane of polarization in the same or different directions. This formula is found to hold good not only with solid crystals, but also with liquids, possessing the property of circular polarization, when mixed together.

Elliptical Polarization.

Where light is polarized by reflection from the surfaces of *transparent media*, the entire beam has but one plane of polarization, and that in the plane of reflection. But when *metallic* surfaces are employed, the beam is found to consist of two portions of unequal intensity polarized in opposite planes. These portions are superposed, and the intensity of that polarized in the plane of reflection is always the greater; its excess, according to Sir David Brewster, above that polarized in the opposite plane, varying with the different metals, being least in silver and greatest in galena.

Colors of Thin Plates.

Transparent media, when reduced to very thin films, are found to exhibit colors which vary with the thickness of the film. These are called the *colors of thin plates*, and the easiest way to exhibit them is by means of a soap bubble blown from the end of a quill or the bowl of a common smoking pipe. As the bubble increases in diameter, and the fluid enveloped is reduced in thickness at the top by gradual subsidence toward the bottom, many colored and concentric rings will be seen around the point of least thickness. At this point, the color will be found to change, first appearing white, then passing through blue to perfect blackness, the rings the while dilating till the bubble is destroyed.

The same is true of any other medium, whether gaseous, fluid or solid.

These different colors being exhibited upon the same plate of variable thickness, no single color can be identified with its chemical composition. When of uniform thickness, a single color only will be seen, and this will change as the thickness of the plate changes.

A thin plate is very conveniently formed of air; and for this purpose, let A B (fig. 102), be a plano-convex, and C D a plano-concave lens, placed one upon the other, as represented in the figure. When this arrangement is viewed on either of the

plane faces by reflected light, colors will be seen in the form of concentric circles about the point of contact, which, should the pressure be sufficient, will be totally black. If viewed by transmitted light, rings complementary to the former will appear about the central spot which will now be perfectly white. In *homogeneous* transmitted light, the rings are alternately bright and dark, beginning with the central spot; and by reflected light, dark and bright. They are broadest and have the greatest diameter in the red, and narrowest with least diameter in the violet; the breadths and diameters in the other colors being intermediate and varying in magnitude in the order of the spectrum from red to violet. It is by the superposition of these rings that the different colors appear in common light.

These colors, which are of different orders as regards tint, constitute what is called Newton's scale; and by reflected light, occur as follows, beginning with the central spot.

1*st order*. Black, very faint blue, brilliant white, yellow, orange and red.
2*d order*. Dark violet, blue, yellow-green, bright yellow, crimson and red.
3*d order*. Purple, blue, rich green, fine yellow, pink and crimson.
4*th order*. Dull blue-green, pale yellow-pink, and red.
5*th order*. Pale blue-green, white and pink.
6*th order*. Pale blue-green, pale pink.

7th order. The same as 6th, very faint. The other orders being too faint to be distinguishable.

If the plano-concave lens be replaced by a metallic reflector, and the light be polarized in a plane at right angles to that of incidence, the rings will disappear when the rays are incident under the maximum polarizing angle. Under these circumstances the light will be wholly transmitted at the first surface, but will be reflected of considerable intensity at the second; hence it is inferred that *the first surface is essential to the formation of the rings.* When the metallic reflector is slightly tarnished, a second system of rings has been observed, arising, as is supposed, from the light irregularly reflected in consequence of the tarnish; this proves *the agency of the second surface in producing the rings*, which are in consequence attributed to the interference of the rays reflected at the first surface with those reflected at the second.

Suppose a small beam incident perpendicularly or nearly so, on the first surface of the plate, where the thickness is t. A part will be reflected back along the incident beam, the rest, being transmitted, will traverse the thickness t. At the second surface a part is again reflected, and the reflected portion returning through the thickness t, will emerge at the first surface in the direction of the incident beam, and be superposed on that reflected at this surface. This superposition may

take place according to any of the conditions explained in (105), with respect to the intervals λ. If dissimilar points are superposed, darkness will result, and to this end the difference of route traversed, which is $2t$, must be equal to $\frac{\lambda}{2}$, or some odd multiple of it.

But the portion reflected at the second surface will, in part, be again reflected at the first, and will traverse the thickness t a third time, and emerge below superposed upon the portion first transmitted at the second surface. The difference of route of these portions will also be $2t$, so that they should also give total darkness. Experiment shows, however, that this is not the case, for wherever there is total darkness by reflection, there is a maximum of brightness by transmission. Hence, if interference be the cause of the rings, there must be *half an interval added to or subtracted from the route at each internal reflection.* This will give for the interfering rays, in case of reflected rings, a difference of route expressed by

$$2t + \frac{\lambda}{2};$$

and for the transmitted,

$$2t + \lambda.$$

To ascertain the value of t, at the different

rings, call d the diameter of any one of them, as determined by actual measurement, r and r the radii of the surfaces, v and v', the versed sine of the arcs whose sines are equal to the semi-diameter of the ring in question.

Then, for very small arcs we have, (fig. 102),

$$v = \frac{\left(\frac{d}{2}\right)^2}{2r};$$

and

$$v' = \frac{\left(\frac{d}{2}\right)^2}{2r'};$$

whence

$$t = v' - v = \frac{d^2}{8}\left(\frac{1}{r'} - \frac{1}{r}\right).$$

In this way Newton found the thickness at the brightest part of the first ring after the central black spot, 0,00000561 of an inch. He also found the diameters of the darkest rings to be as the square roots of the even numbers 0 . 2 . 4 . 6 . &c., and those of the brightest as the square roots of the odd numbers 1 . 3 . 5 . 7. &c. The radii of the surfaces being great compared with the diameters of the rings, the value of t at the alternate points of

greatest obscurity and illumination are as the natural numbers

$$0 . 1 . 2 . 3 . 4 . \&c.,$$

hence, the value of t, just found, multiplied by these numbers will give the thickness at the different rings.

On comparing the value for the thickness at the first bright ring, with the numbers in the table of article (106), it will be found just equal to one fourth of the interval denoted by λ, for the yellow ray, which is the most illuminating of the elements of white light.

Taking this value for t, we shall have for the difference of route of the interfering rays producing the dark rings by reflection, including the central black spot,

$$\frac{\lambda}{2}, \frac{3\lambda}{2}, \frac{5\lambda}{2}, \frac{7\lambda}{2}, \&c.,$$

these being the even multiples of $\frac{\lambda}{4}$, increased by $\frac{\lambda}{2}$, for the acceleration or retardation by one internal reflection.

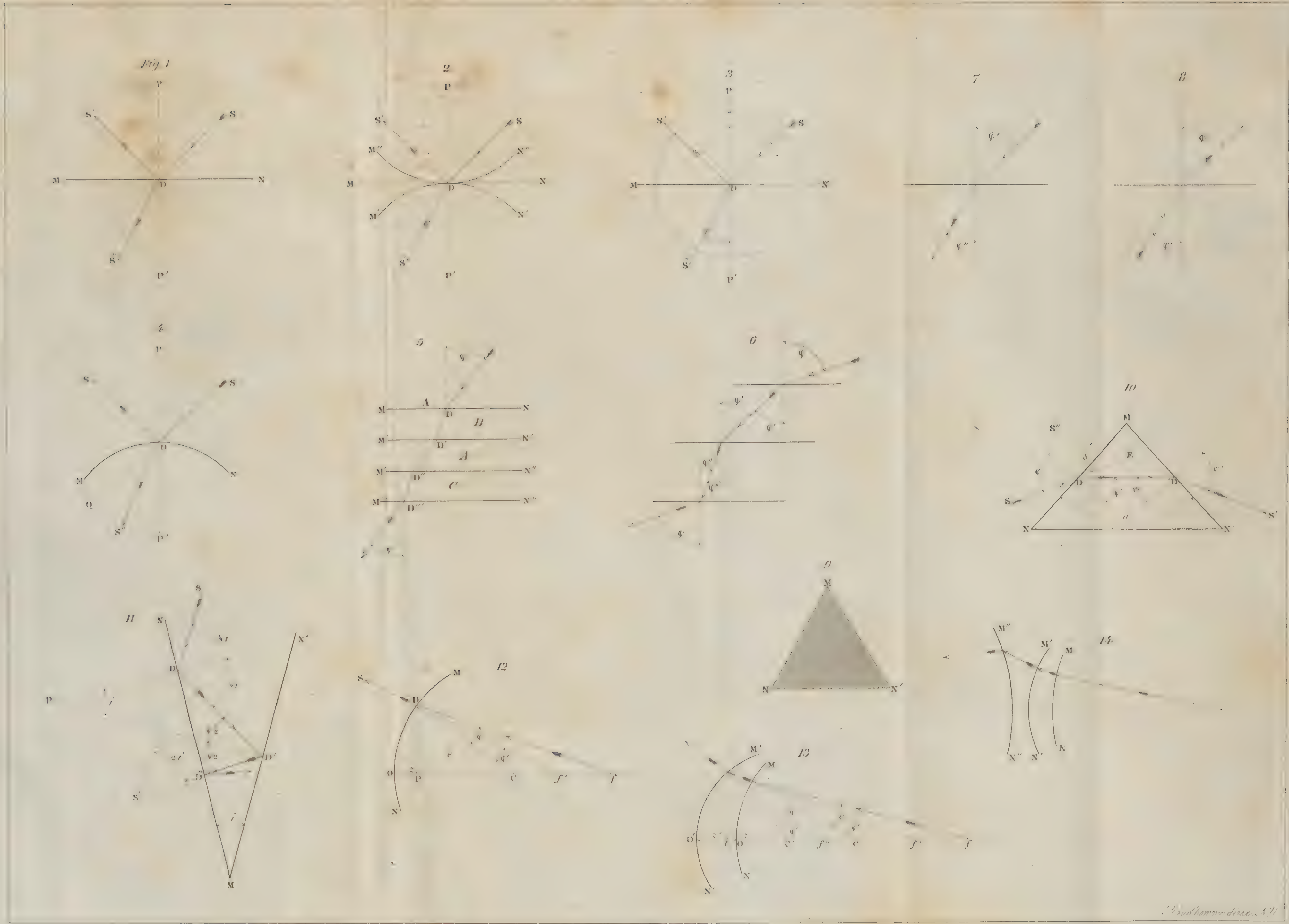
The odd multiples give

$$\lambda, 2\lambda, 3\lambda, \&c., \&c.,$$

from which it is obvious, that the transmitted rays will be complementary to those seen by reflection.

The phenomena we have just considered are equally produced, whatever may be the medium interposed between the glasses, the only difference being in the contraction or expansion of the rings depending upon the refractive power of the medium. It is found that as the refractive power of the medium increases, the diameter of the rings will decrease, which might have been inferred from article (107).

THE END.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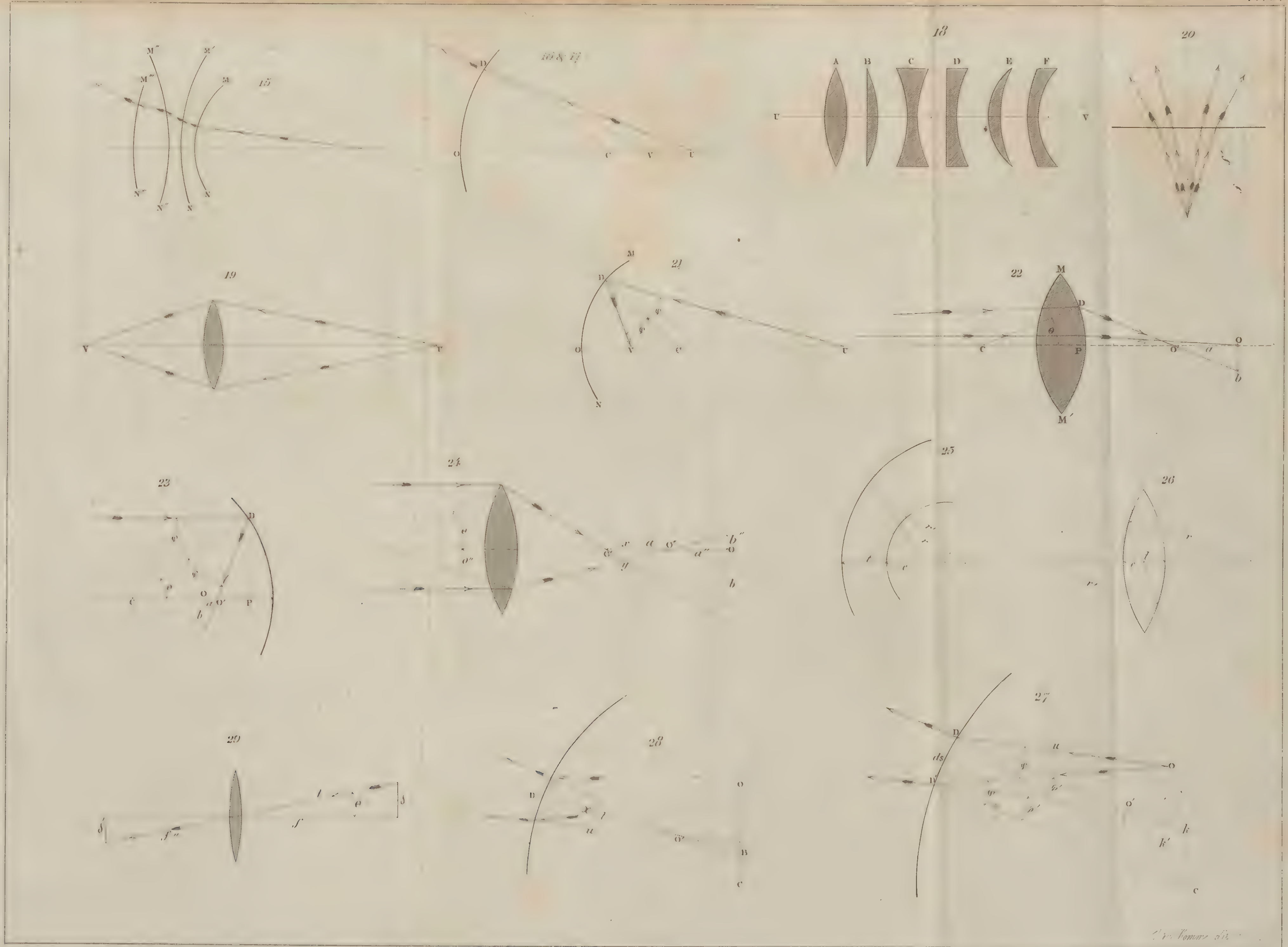
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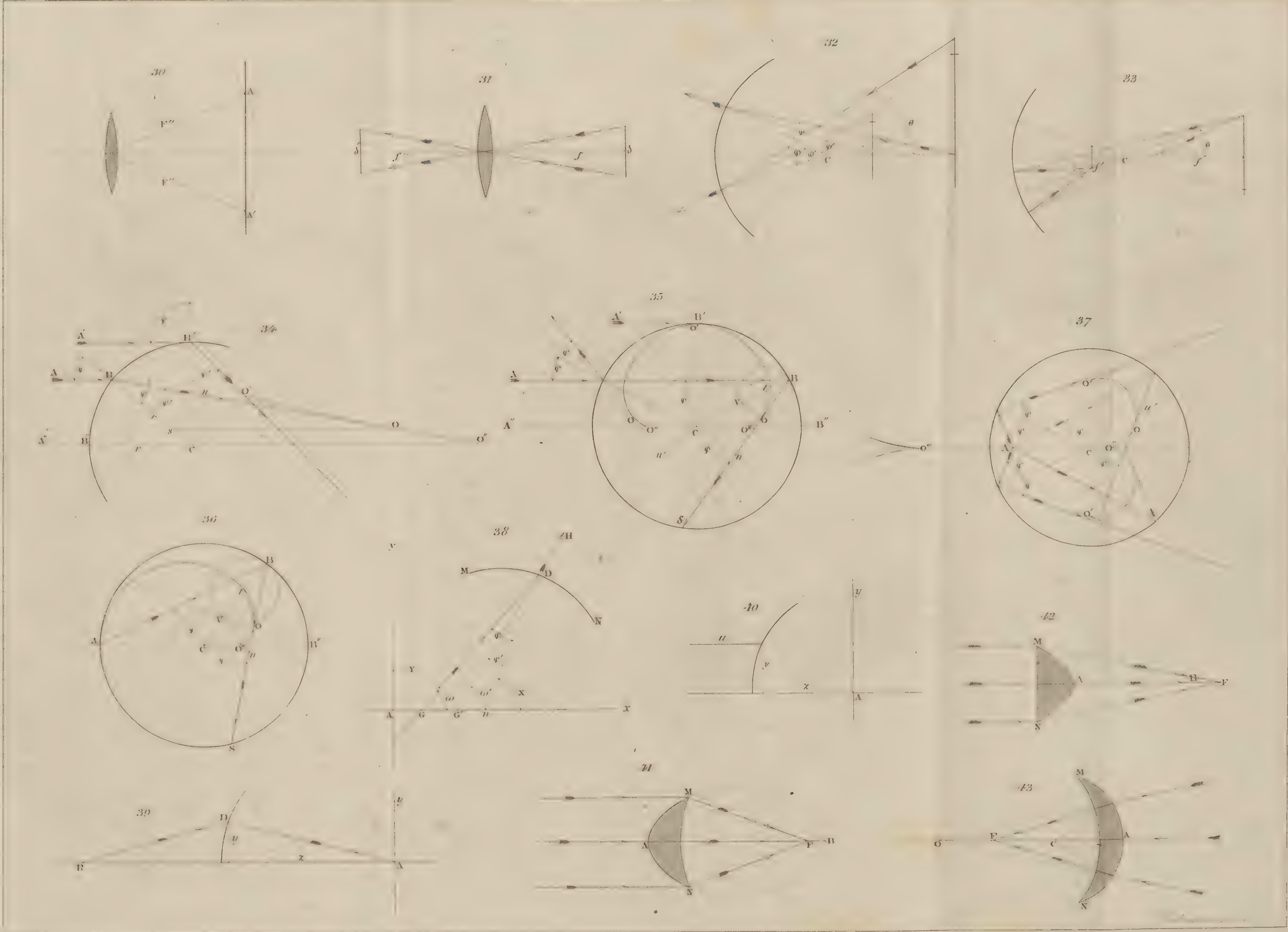
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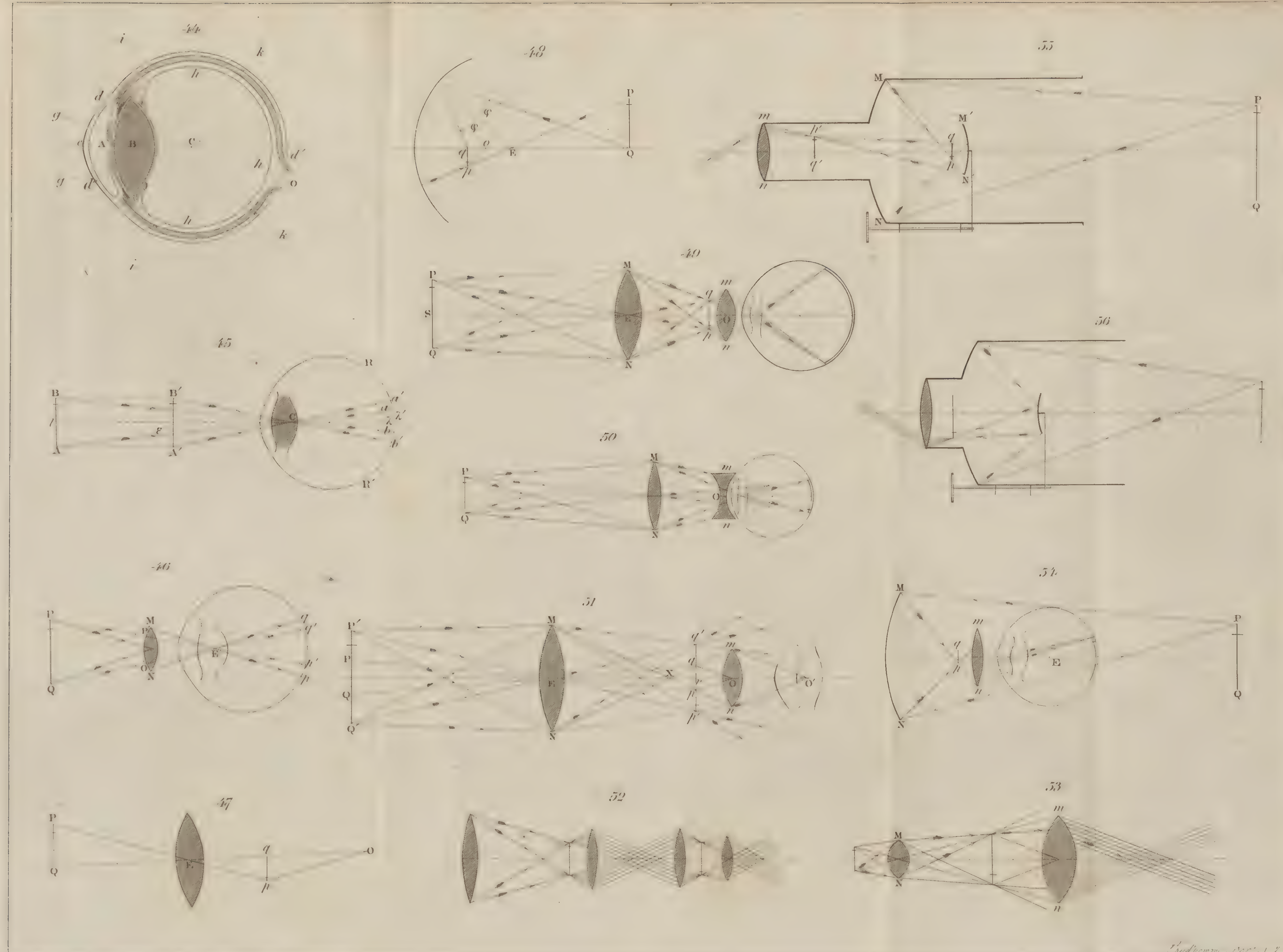
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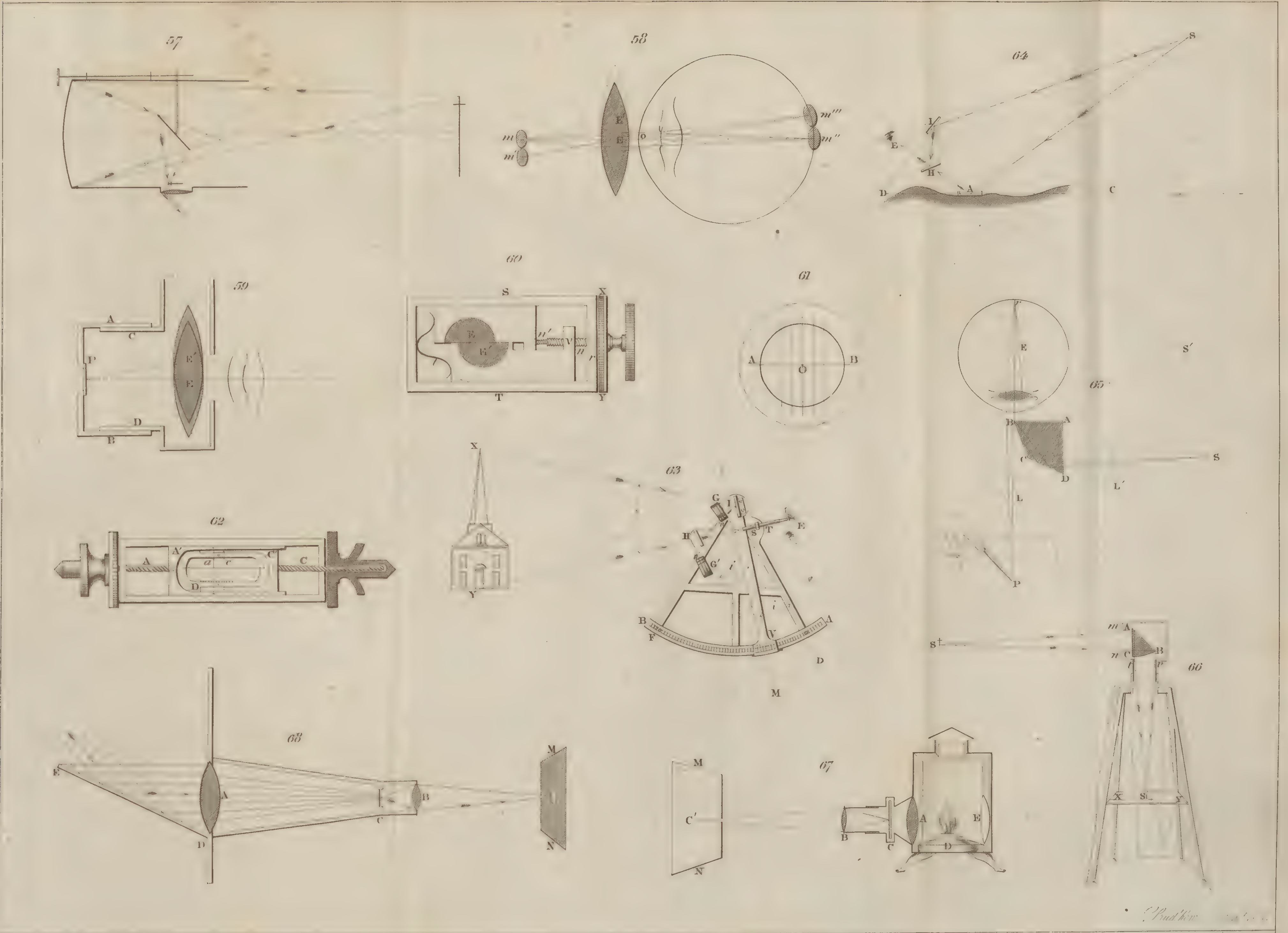
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Crown Glass
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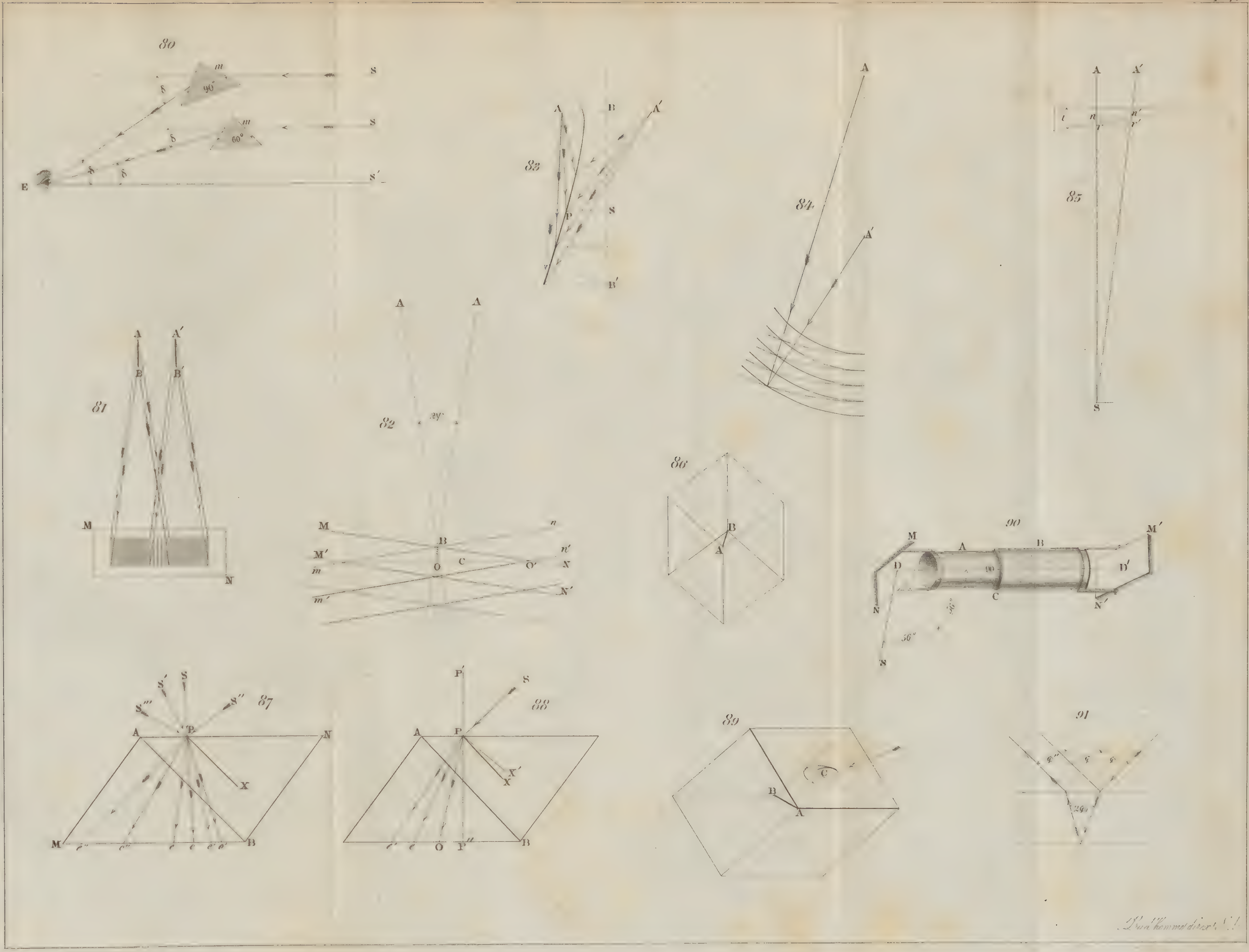
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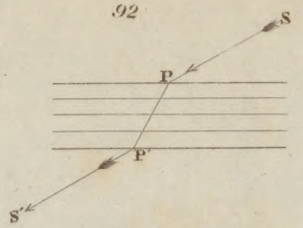

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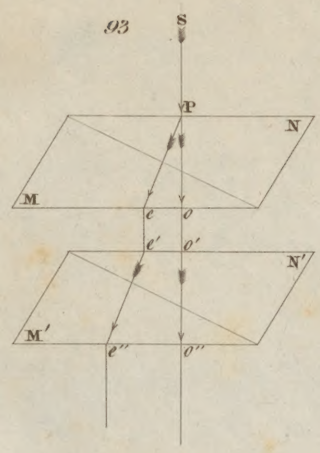

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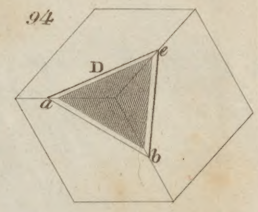

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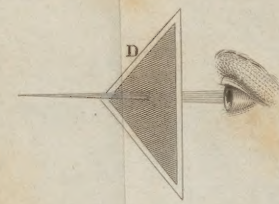

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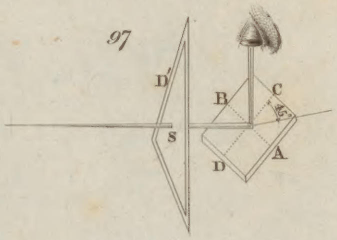

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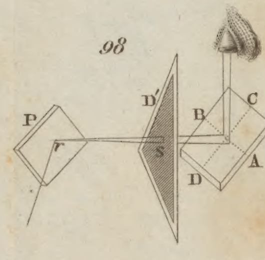

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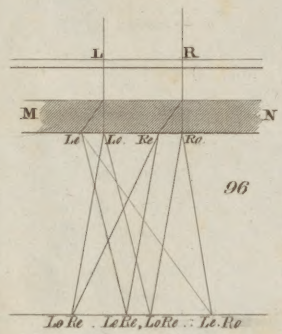

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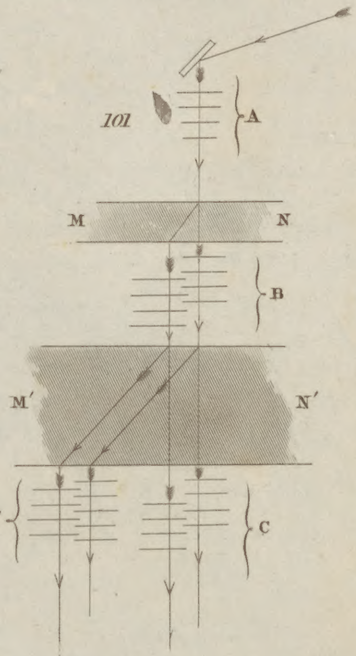

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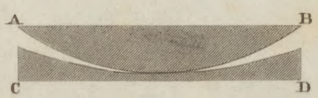

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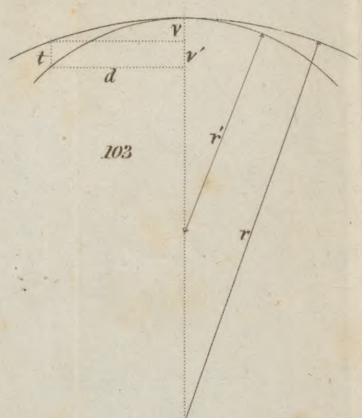

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Prudhomme direx N.Y.